SYMMETRIZABLE MARKOV MATRICES

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Introduction. Suppose that the evolution of the state probabilities $p_i(t)$ of a Markov process is governed by the system of differential equations

$$\frac{dp_i}{dt} = \sum_{j=1}^{N} Q_{ij} p_j(t), \qquad i = 1, \cdots, N, \tag{1}$$

where Q_{ij} represents the transition rate from state S_j to state S_i ([1], p. 235). In many applications one is interested primarily in knowing the equilibrium state probabilities π_i defined by $\pi_i = \lim_{t \to \infty} p_i(t)$, which, if they exist, can be obtained by solving the system of homogeneous linear equations

$$\sum_{j=1}^{N} Q_{ij} \pi_j = 0, \qquad i = 1, 2, \cdots, N. \tag{2}$$

While (2) can be solved in principle (and in practice if N is not too large), the solution in general does not fulfill the ultimate desideratum of being susceptible to representation as a simple function of the transition rates Q_{ij}. If the states are simply ordered and a transition from a given state S_i can occur only to a neighboring state S_{i-1} or S_{i+1} then the equilibrium probability π_i satisfies the following simple formula

$$\pi_i = \frac{\lambda_1 \lambda_2 \cdots \lambda_i}{\mu_2 \mu_3 \cdots \mu_{i+1}} \pi_1, \tag{3}$$

where λ_k is the transition rate from state S_k to state S_{k+1} and μ_k, the transition rate from state S_{k+1} to state S_k, and π_1 is chosen so that

$$\sum_{i=1}^{N} \pi_i = 1. \tag{4}$$

Processes of this sort, with simply ordered sets of states, are called birth and death processes. The discussion of Section 1 below deals with a class of Markov matrices Q which includes the set of birth and death matrices and allows representations analogous to (3) for the equilibrium probabilities. In Section 2 it is shown that all the matrices in this class have non-positive characteristic values and in consequence of this fact, the difference between the state probability $p_i(t)$ and the corresponding equilibrium probability π_i is majorized by a function of t and the set of initial state probabilities $p_i(0)$. In Section 3 the foregoing theory is illustrated by an anisotropic random walk.

The following notions and notations are used. Let Q be an $N \times N$ matrix. The graph $G(Q)$ associated with Q consists of vertices $V_1, V_2, \cdots, V_N$ and of

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