THE SUPREMUM AND INFIMUM OF THE POISSON PROCESS[1]

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1. Introduction. Let $\{X(t); t \geqq 0\}$ be a separable Poisson process with shift such that

$$\log E(e^{i\omega X(t)}) = -it\omega\alpha + \lambda t(e^{i\omega} - 1) \tag{1}$$

for all real ω, and $\alpha, \lambda > 0$. Set

$$\sigma(x, T) = P[\sup_{0 \leqq t \leqq T} X(t) \leqq x].$$

The task of obtaining $\sigma(x, T)$ explicitly for general stochastic processes is intrinsically difficult. However, Baxter and Donsker [1], following the methods and results of Spitzer, have obtained the double Laplace transform of $\sigma(x, T)$ for processes with stationary and independent increments. Their result as it pertains to the Poisson process is as follows.

Theorem. *Let $\{X(t); t \geqq 0\}$ be a separable process satisfying $X(0) = 0$ a.s and*

$$\log E(e^{i\omega X(t)}) = t\psi(\omega)$$

for all $t \geqq 0$ where exp $(\psi(\omega))$ *is the Lévy-Khintchine representation of the characteristic function of an infinitely divisible distribution. If $\psi(\omega)$ is complex and for some $\delta > 0$,*

$$\int_{-\delta}^{\delta} \left| \frac{\psi(\omega)}{\omega} \right| d\omega < \infty,$$

then for all $u, v \geqq 0$,

$$\begin{aligned} u \int_0^\infty \int_{0-}^\infty e^{-uT-vx}\, d_x\, \sigma(x, T)\, dT \\ = \exp\left\{\frac{1}{2\pi} \int_u^\infty \int_{-\infty}^\infty \frac{v}{\omega(\omega - iv)} \frac{\psi(\omega)}{s[s - \psi(\omega)]}\, d\omega\, ds\right\}. \end{aligned} \tag{2}$$

Theoretically, therefore, to obtain $\sigma(x, T)$ explicitly, one should evaluate the double integral on the right hand side of (2) and then perform a double inversion on it. For most cases this is virtually impossible except by numerical methods. Baxter and Donsker, however, have evaluated the right hand side of (2) for several important cases. Moreover, for the Gaussian process and for the process determined by coin tossing at random times, they were able to make the inversions.

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