INTUITIVE PROBABILITY ON FINITE SETS[1]

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1. Introduction. Let $x_1, \cdots, x_n$ be the distinct elements of a set S. By assigning nonnegative numbers $v(x_i)$ to the x_i and $v(x_{i_1}) + \cdots + v(x_{i_s})$ to the set $\{x_{i_1}, \cdots, x_{i_s}\}$, we obtain an ordering of the subsets of S, namely, the subsets are ordered in accordance with the values as just assigned.[2] We denote by $v(\alpha)$ the value assigned to α, and write $\alpha \prec \beta$ if $v(\alpha) \leqq v(\beta)$. For this ordering the following conditions obtain:

Comparability (C): For any α, β, $\alpha \prec \beta$ or $\beta \prec \alpha$ (or both).

Transitivity (T): $\alpha \prec \beta$ and $\beta \prec \gamma$ implies $\alpha \prec \gamma$

Additivity (A): Let γ be disjoint from α, β; then $\alpha \prec \beta$ if and only if

$$\alpha \cup \gamma \prec \beta \cup \gamma.$$

Also $\phi \prec \gamma$ for every γ, where ϕ is the empty set.

Let T be the set of subsets of S. We shall say that an ordering of T obtained by assigning values to the x_i *arises from a measure*. Conversely, B. de Finetti [1] (see also [4], p. 40) has asked whether every ordering of T subject to the above conditions arises from a measure; and moreover has conjectured that it does; but we show by a counter-example that the conjecture is false for $n = 5$. In Theorem 2 we give a necessary and sufficient condition that an ordering arises from a measure; the proof includes a procedure for checking in a finite number of steps whether the condition obtains.

The connection with intuitive probability (i.e., the axiomatic theory of probability) is as follows: one has n incompatible events $x_1, \cdots, x_n$; and one supposes that one can confront the disjunction of any subset of them with the disjunction of any other, being able to say (or judge) whether they are equally likely, and if not, which is the more likely. Thus one has a transitive ordering of T; moreover, this ordering is subject to the additivity condition (and, if one likes, to any further conditions similar to the above which obtain for an ordering arising from a measure). The question then is whether one can assign a numerical probability to the event x_i in such a way that the corresponding ordering of T coincides with the given ordering; or in other words, whether there exists a *strictly agreeing measure*. As said, the answer is *no*.

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[2] By an *ordering* of a set S we mean an arbitrary, possibly empty, subset of the Cartesian product $S \times S$, that is, an arbitrary set of ordered pairs (a, b) with a, b elements of S. If (a, b) is such a pair, we write $a \prec b$. An ordering is sometimes also called a relation.