BOUNDS ON THE EXPECTATION OF A CONVEX FUNCTION OF A MULTIVARIATE RANDOM VARIABLE

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1. Introduction. Dresher has shown [2] how certain inequalities can be interpreted geometrically via the theory of moment spaces of univariate distributions. Moment spaces of multivariate distributions will be considered, and, by examining the boundary of an appropriate moment space, upper and lower bounds on the expectation of a convex function of a vector valued random variable will be derived. Finally, the bounds so derived will be improved in the case where the elements of the random vector are independent.

2. Moment spaces of multivariate distributions. Let $\psi(x) = \psi(x_1, \cdots, x_r)$ be an r-variate cumulative distribution function over the bounded r-dimensional rectangle I, and let $\{f_i(x_1, \cdots, x_r) = f_i(x), i = 1, \cdots, n\}$ be a set of n continuous functions. The ith moment of $\psi(x)$ with respect to $\{f_i(x)\}$ is defined to be $\mu_i(\psi) = \int_I f_i(x)\, d\psi(x)$, and the nth moment space M_n with respect to $\{f_i(x)\}$ is defined as the set of all points $\mu = (\mu_1, \cdots, \mu_n)$ in n-dimensional Euclidean space, E_n, whose coordinates are the moments $\mu_1(\psi), \cdots, \mu_n(\psi)$ with respect to $\{f_i(x)\}$ for some distribution function $\psi(x)$.

Let C_n be the surface traced out in E_n by

$$\{z_i = f_i(x), i = 1, \cdots, n, x \varepsilon I\}.$$

Let H_n be the convex hull of C_n, i.e., the smallest convex set containing C_n. Then it can be shown, along the same lines as the proof of Theorem 2 of [1], that H_n is identical with M_n, and that M_n is closed, bounded, and convex.

In the following I shall examine $g(X)$, some given continuous convex function of an r-dimensional vector valued random variable X defined over the bounded r-dimensional rectangle I. Let C_{r+1} be the surface traced out in E_{r+1} by $z_1 = x_1, z_2 = x_2, \cdots, z_r = x_r, z_{r+1} = g(x_1, \cdots, x_r) = g(x)$. What I shall do is determine the boundary of H_{r+1}, the convex hull of C_{r+1}, from which inequalities on $Eg(X)$ in terms of $g(EX)$ and $\{EX_i, i = 1, \cdots, r\}$ will be obtained. A point x^0 is said to be on the boundary of H_{r+1} if and only if x^0 is in H_{r+1} and there exists a set of real numbers $\beta_0, \beta_1, \cdots, \beta_{r+1}$ such that

$$\sum_{i=1}^{r+1} \beta_i x_i^0 + \beta_0 = 0$$

and

$$\sum_{i=1}^{r+1} \beta_i x_i + \beta_0 \geqq 0 \qquad \text{for all } x \text{ in I.}$$

Geometrically, x^0 is on the boundary of H_{r+1} if and only if there exists a supporting hyperplane to H_{r+1} at that point. A point x^0 is on the upper (lower)

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