ON A CHARACTERIZATION OF COVARIANCES

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1. Introduction. Let $F(s, t)$, $-\infty < s, t < \infty$, be a covariance function, that is to say $F(s, t) = \overline{F(t, s)}$ and $F(s, t)$ is non-negative definite. Let $m(s)$ be any complex valued function on $-\infty < s < \infty$. It is trivial that then

$$F(s, t) + m(s)\overline{m(t)}$$

is also a covariance. However, this is no longer true if we consider instead

$$F(s, t) - m(s)\overline{m(t)}. \tag{1}$$

In this paper we obtain a set of necessary and sufficient conditions on $m(s)$ in order that (1) be a covariance under the restriction that $F(s, t)$ is a stationary covariance; i.e., $F(s, t) = F(s - t)$. We also indicate an application of the result to the problem of estimating the mean value of a stochastic process.

2. Main results.

THEOREM 1. *Let* $R(t)$ *be a continuous stationary covariance function with* $R(0)$ *finite. Let* $m(s)$ *be any function on* $-\infty < s < \infty$. *Then a necessary and sufficient condition that*

$$R(s, t) = R(s - t) - m(s)\overline{m(t)} \tag{2}$$

be a covariance function is that $m(t)$ *have the representation*

$$m(t) = \int_{-\infty}^{\infty} \exp(itx)\, d\mu, \tag{3}$$

where $\mu(\cdot)$ *is a function of bounded variation, and that, further,*

$$\int_{-\infty}^{\infty} |d\mu/dG|^2\, dG \leqq 1, \tag{4}$$

$G(\cdot)$ *being the spectral distribution corresponding to* $R(t)$, *so that*

$$R(t) = \int_{-\infty}^{\infty} \exp(itx)\, dG. \tag{5}$$

PROOF. *Necessity:* Let $R(s, t)$ given by (2) be a covariance. Then we can (see [1], p. 72) construct a Gaussian process $y(t)$, $-\infty < t < \infty$, with zero mean so that $E[y(s)\overline{y(t)}] = R(s, t)$. Now, since $R(t, t)$ must be non-negative if (2) is to yield a covariance, $m(t)$ is necessarily bounded. Letting

$$x(t) = y(t) + m(t),$$

we have $E[x(s)\overline{x(t)}] = R(s - t)$, so that the $x(t)$ process has finite first and second moments and is stationary in the wide sense. Moreover, $R(t)$ is continuous. Using the spectral representation theorem ([1], p. 527), we have

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