NON-MARKOVIAN PROCESSES WITH THE SEMIGROUP PROPERTY

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1. Introduction. Every $N \times N$ stochastic matrix P defines the transition probabilities of a Markovian process with positive discrete time parameter. Its n-step transition probabilities satisfy the Chapman-Kolmogorov, or semigroup, relation $P^{n+m} = P^n P^m$. We shall show that *for $N \geqq 3$ there exist non-Markovian processes with N states whose transition probabilities satisfy the same equation.*[2] All elements of P will equal N^{-1}. The process may be chosen *strictly stationary*. A simple modification leads to non-Markovian processes with *continuous time* parameter and the semigroup property with N states or a continuum of states.

The triviality of the following example should not obscure the interest of the problem concerning the existence of non-Markovian processes satisfying the Chapman-Kolmogorov equation. As so many other basic problems in probability theory, it has been formulated by P. Lévy who with his usual ingenuity gave the first counter-example to the obvious conjecture.

2. Let $\mathfrak{P}$ be the sample space whose points $(x^{(1)}, \cdots, x^{(N)})$ are the random permutations of $(1, 2, \cdots, N)$ each carrying probability $1/N!$ Let $\mathfrak{R}$ be the set of the N points $(x^{(1)}, \cdots, x^{(N)})$ such that $x^{(i)} = \nu$ for all $1 \leqq i \leqq N$ where ν is a fixed integer $1 \leqq \nu \leqq N$; each point of $\mathfrak{R}$ carries probability $1/N$. Finally, Let $\mathfrak{S}$ be the mixture of $\mathfrak{P}$ and $\mathfrak{R}$ with $\mathfrak{P}$ carrying weight $1 - N^{-1}$ and $\mathfrak{R}$ weight N^{-1}.

More formally, $\mathfrak{S}$ contains the $N! + N$ arrangements $(x^{(1)}, x^{(2)}, \cdots, x^{(N)})$ which represent either a premutation of $(1, 2, \cdots, N)$ or the N-fold repetition of an integer ν, $1 \leqq \nu \leqq N$. To each point of the first class we attribute probability $(1 - N^{-1})(N!)^{-1}$, to each point of the second class probability N^{-2}.

Then clearly

$$(1) \qquad P\{x^{(i)} = \nu\} = N^{-1}, \qquad P\{x^{(i)} = \nu, \quad x^{(j)} = \mu\} = N^{-2}$$

for all $i \neq j$. Thus all transition probabilities are equal:

$$(2) \qquad P\{x^{(i)} = \nu \mid x^{(j)} = \mu\} = N^{-1}.$$

Given, say, that $x^{(1)} = 1$, $x^{(2)} = 1$ the probability that $x^{(3)} \neq 1$ is zero, and hence the process is non-Markovian.

3. To extend the process to all integral values of the time parameter consider, in the usual manner, a double infinity of independent repetitions of the described

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[2] [Added in proof.] D. Blackwell has pointed out to me that the variables of our process represent a sequence of *random variables which are pairwise independent without being mutually independent.*