# LARGE EXCURSIONS OF GAUSSIAN PROCESSES

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**1. Introduction.** It is known that the problem of determining the distribution of spacings between consecutive $a$-values of an ergodic Gaussian process, $x(t)$, $(Ex(t) = 0,\ Ex^2(t) = 1)$ is very difficult. Recently Palmer [1] and Rice [2] treated some limiting cases of this problem. In one limit they determine, for $a \to \infty$, the conditional probability

$$\Pr\{x(\tau) > a, \quad 0 \leqq \tau \leqq t\theta(a) \mid x(0) = a, \quad x'(0) > 0\} \tag{1.1}$$

where $\theta(a)$ is the average length of the times spent by $x(t)$ above the level $a$. Apart from some differences concerning the meaning of the conditional probability (1.1) both authors use the following heuristic device.

Since for large $a$, $\theta(a)$ is small, they write

$$x(\tau) = a + x'(0)\,\tau + \frac{x''(0)}{2}\,\tau^2 \tag{1.2}$$

and take for the time of the first downward crossing of the $a$-level

$$\tau = -2\,\frac{x'(0)}{x''(0)}\,. \tag{1.3}$$

It would thus seem that this procedure is limited to processes for which $x''$ exists. This would exclude, for example, the displacement of a harmonic oscillator in Brownian motion. It is precisely this point that led us to undertake the present investigation.

We have found an alternative derivation of the Palmer-Rice results which does not depend on the approximation (1.2) and hence is applicable to all cases of physical interest. We have also attempted to elucidate the ambiguity of (1.1) (see §2) and we have in §3 shown in what sense the sample functions $x(\tau)$ are approximated by parabolas as suggested in (1.2).

**2. Conditional probability densities.** It is well known that conditional probabilities and conditional probability densities must frequently be treated with some care. Since the material to follow contains some excellent examples of the subtle nature of these quantities, a few words on the subject are in order here.

Let $x(t)$ be a continuous ergodic Gaussian process possessing a derivative almost everywhere. Consider the "conditional probability density for the slope $\xi = x'(0)$ given that $x(0) = a$." From the ensemble point of view, the phrase in quotation marks has no meaning, since the set of sample functions satisfying the condition $x(0) = a$ is of probability zero. Yet, given a sample function of

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