RANDOM GRAPHS

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1. Introduction. Let N points, numbered $1, 2, \cdots, N$, be given. There are $N(N-1)/2$ lines which can be drawn joining pairs of these points. Choosing a subset of these lines to draw, one obtains a graph; there are $2^{N(N-1)/2}$ possible graphs in total. Pick one of these graphs by the following random process. For all pairs of points make random choices, independent of each other, whether or not to join the points of the pair by a line. Let the common probability of joining be p. Equivalently, one may erase lines, with common probability $q = 1 - p$ from the complete graph.

In the random graph so constructed one says that *point i is connected to point j* if some of the lines of the graph form a path from i to j. If i is connected to j for every pair i, j, then the graph is said to be *connected*. The probability P_N that the graph is connected, and also the probability R_N that two specific points, say 1 and 2, are connected, will both be found.

As an application, imagine the N points to be N telephone central offices and suppose that each pair of offices has the same probability p that there is an idle direct line between them. Suppose further that a new call between two offices can be routed via other offices if necessary. Then R_N is the probability that there is some way of routing a new call from office 1 to office 2 and P_N is the probability that each office can call every other office.

Exact expressions for P_N and R_N are given in Section 2. These results are unwieldy for large N. Bounds on P_N and R_N derived in Section 3 show that

$$P_N \sim 1 - Nq^{N-1} \tag{1}$$

and

$$R_N \sim 1 - 2q^{N-1} \tag{2}$$

asymptotically as $N \to \infty$.

Other related results appear in a paper by Austin, Fagen, Penney, and Riordan [1]. These authors use a different random process to pick a graph and they find a generating function for the distribution of the number of connected pieces in the random graph.

2. Exact results. P_N may be expressed in terms of the number $C_{N,L}$ of connected graphs having N labeled points and L lines. Since each such graph has probability $p^L q^{-L+N(N-1)/2}$ of being the chosen graph, it follows that

$$P_N = \sum_L C_{N,L} p^L q^{-L+N(N-1)/2}.$$

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