AN EXAMPLE OF WIDE DISCREPANCY BETWEEN FIDUCIAL AND CONFIDENCE INTERVALS

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1. Introduction. Fisher [1], [2] has emphasized that when he chooses a set in the parameter space on the basis of certain observations and attributes to it a certain fiducial probability α, he does not intend that, for fixed values of the parameter the probability that this random set contains the parameter point should be α. Examples of this distinction for the Behrens-Fisher problem have been given by Fisher [1], [2] and Neyman [3], [4]. In these cases the numerical differences are not extremely large. In order to bring out more clearly the contrast between fiducial probability and confidence sets I shall give, for each α and ϵ in the interval $(0, 1)$, an example where a fiducial interval for a parameter with fiducial probability equal to α has probability less than ϵ of covering the true parameter for a large range of parameter values. This means that although a large fiducial probability is claimed, it is practically certain that the interval will not cover the true parameter value. Of course this cannot happen when the fiducial sets are obtained by Pitman's methods [6], [7].

2. The example. Let $X_1, \cdots, X_n$ be independently normally distributed real random variables with unknown means $\xi_1, \cdots, \xi_n$ and variance 1. Suppose we are interested in fiducial or confidence sets for $\sum \xi_i^2$ of the form

$$[f(X_1, \cdots, X_n), \infty].$$

We consider the one-sided case only in order to avoid irrelevant computational details. The fiducial distribution of $\xi_1, \cdots, \xi_n$ is that they are independently normally distributed with means $X_1, \cdots, X_n$ and variance 1 (see Fisher [1], p. 132, where the case $n = 2$ is given, but see also Tukey [5] for a different fiducial distribution). Thus the fiducial distribution of $\sum_1^n \xi_i^2$ is a non-central χ^2 distribution with n degrees of freedom and non-centrality parameter $\sum_1^n X_i^2$. On the basis of this we determine a fiducial interval

$$[\Phi_{\alpha,n}(\textstyle\sum X_i^2), \infty) \tag{1}$$

with fiducial probability α for the unknown parameter $\sum \xi_i^2$. Here $\Phi_{\alpha,n}(\sum X_i^2)$ is the value which will be exceeded with probability α by a non-central χ^2 variate with n degrees of freedom and non-centrality parameter $\sum X_i^2$. But this non-central χ^2 distribution is, for large n, approximately a normal distribution with mean $n + \sum X_i^2$ and variance $2n + 4\sum X_i^2$, the approximation being uniform

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