A GEOMETRY OF BINARY SEQUENCES ASSOCIATED WITH GROUP ALPHABETS IN INFORMATION THEORY[1]

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1. The group alphabet. When a piece of information, or *letter* is transmitted over a symmetric binary channel [14], the letter is presented to the channel in the form of a sequence of n binary digits. Because of *noise* on the channel, there is a positive probability p that a transmitted symbol will be received in error, that is, a transmitted 0 will be received as 1 or transmitted 1 received as 0. It is assumed that $0 < p < \frac{1}{2}$, and that the noise on the channel operates independently on each symbol that is presented for transmission. If the collection of all distinct pieces of information—the *alphabet*—consists of $K = 2^k$ letters, it is customary to take $n > k$, and in some manner use the additional digit positions to "correct" errors in transmission. Slepian [14] has introduced the *n-place group alphabet*, or, briefly, the *(n, k)-alphabet*, and an associated decoding scheme. The 2^n possible binary sequences form an Abelian group B_n wherein the group operation is addition modulo 2 of vectors given by the sequences. An (n, k)-alphabet is a 2^k-letter n-place binary signaling alphabet whose letters form a subgroup of B_n.

Let us designate the letters of the alphabet by

$$U_0 = I = (000 \cdots 0), U_1, U_2, \cdots, U_\mu, \mu = 2^k - 1.$$

The group B_n can be developed according to the alphabet and its cosets:

$$\begin{array}{lllll} I = U_0 = L_0, & U_1, & U_2, & \cdots & U_\mu, \\ L_1 \quad, & L_1 + U_1, & L_1 + U_2, & \cdots & L_1 + U_\mu, \\ L_2 \quad, & L_2 + U_1, & L_2 + U_2, & \cdots & L_2 + U_\mu, \\ \cdots & \cdots & \cdots & & \cdots \\ L_\nu \quad, & L_\nu + U_1, & L_\nu + U_2, & \cdots & L_\nu + U_\mu, \end{array} \tag{1}$$

where $\nu = 2^{n-k} - 1$, and L_l is an n-place binary sequence which has not appeared in cosets led by $L_0, L_1, \cdots, L_{l-1}$. The group elements L_l are called *coset leaders*.

The *weight* $w(T_j)$ of an element T_j of B_n is defined as *the number of ones in the n-place binary sequence* T_j.

Because of the group property, any coset is repeated, with elements in a

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