ON TIME DEPENDENT QUEUING PROCESSES

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1. Introduction. It is well known that the general class of stochastic processes with discrete states in continuous time arising in queuing theory, birth-death processes, etc., can be characterized as Markov processes provided the full set of random variables needed to specify the state of the process is employed. A detailed illustration of this approach is given by Cox in [1][1] for the case of a queue in equilibrium subject to a random (Poisson) arrival distribution and a general service time distribution. Our object in this paper is to initiate a systematic development of this approach in the theory of queues. It turns out that such a development for the time dependent version of the above described queuing problem requires analytical considerations not encountered in the equilibrium case. Similarly the systematic development of this approach for the queue with a general arrival distribution (as well as a general service time distribution) leads to a still different type of mathematical problem (simultaneous Wiener-Hopf integral equations with an analytic side condition) which we intend to report on elsewhere.

One final remark is in order concerning the formulation of the approach and derivation of the governing differential equations carried out in sections 2 and 3. While there is a general similarity between the arguments in these sections and those, for example, in [1], we prefer to give a self contained discussion in order to exhibit how the additional complications arising from the consideration of time dependence can be incorporated in the general approach.

2. Phase space. We assume a Poisson arrival distribution with a mean rate of arrival λ and that the service time x between an admission and completion is specified by an arbitrary probability density $D(x)$.

The state of the entire system (queue and service operation) at time t is specified by the number, m, of people in queue and the elapsed time, x, of the person currently in service. Our phase space Γ, accordingly, will be two dimensional with one discrete dimension consisting of the non-negative integers (queue lengths) and one continuous dimension consisting of the positive reals x. The state of the system is then characterized by a point in Γ. For completeness there should be a single additional point in Γ corresponding to the state of total vacancy of the system.

We can now introduce the probability density $W_m(x, t)$ on Γ for the probability that at time t the queue length, excluding servee, is m and the elapsed time in service is x. It is worth emphasizing that the characterization of the state of the system by means of the set of probability densities $W_m(x, t)$ is

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[1] We are indebted to the referee for bringing Cox's work to our attention.