THE FIRST-PASSAGE MOMENTS AND THE INVARIANT MEASURE OF A MARKOV CHAIN

By John Lamperti[1]

Stanford University

We consider an irreducible, recurrent Markov chain with transition probability matrix $P = [p_{ij}]$. The random variables constituting the chain are $\{X_n\}$; let $N > 0$ be the smallest positive time n at which $X_n = 0$. Then the quantities

$$E\{N(N-1)\cdots(N-k+1) \mid X_0 = i \neq 0\} = \mu_{i0}^{(k)}$$

are the factorial first-passage time moments. In case $i = 0$, we will let $\mu_{00}^{(k)} = \delta_{0k}$. However, it is also convenient to introduce the actual recurrence-time moments for state 0:

$$E\{N(N-1)\cdots(N-k+1) \mid X_0 = 0\} = \mu_0^{*(k)}.$$

Let $\{\pi_i\}$ be the unique positive solution of the equation

$$\pi_j = \sum_i \pi_i p_{ij}, \tag{1}$$

often called the "invariant measure" of the chain. Then this measure and the first-passage moments are related by the

Theorem. *The equation*

$$\pi_0 \mu_0^{*(k+1)} = (k+1) \sum_i \pi_i \mu_{i0}^{(k)}, \qquad k = 0, 1, 2, \cdots, \tag{2}$$

is always valid. (*Both sides may be* $+\infty$.)

Remarks. If $k = 0$, (2) reduces to the familiar assertion that the mean recurrence time of state 0 is $\pi_0^{-1}\Sigma\pi_i$. If $k = 1$, (2) is equivalent to a "remarkable formula" discovered by Chung [1], who gave a proof rather different from that which follows.

Proof of the Theorem. We shall use generating functions; let

$$f_{i0}^{(n)} = \Pr\{X_n = 0, X_l \neq 0 \text{ for } l < n \mid X_0 = i \neq 0\}$$
$$= \Pr\{N = n \mid X_0 = i \neq 0\}; \qquad f_{00}^{(n)} = \delta_{n0}; \qquad F_{i0}(x) = \sum_{n=0}^{\infty} f_{i0}^{(n)} x^n.$$

Thus $F_{00}(x) = 1$, and $F_{i0}^{(k)}(1) = \mu_{i0}^{(k)}$ for all i including 0. Similarly we put $g_0^{(n)} = \Pr\{X_n = 0, X_l \neq 0 \text{ for } 1 \leqq l < n \mid X_0 = 0\} = \Pr\{N = n \mid X_0 = 0\}$, and $G_0(x) = \sum_{n=1}^{\infty} g_0^{(n)} x^n$. Notice that $g_0^{(n)} = \sum_i p_{0i} f_{i0}^{(n-1)}$, so that

$$G_0(x) = x \sum_i p_{0i} F_{i0}(x). \tag{3}$$

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