ON A PROBLEM OF J. NEYMAN AND E. SCOTT[1]

By L. Schmetterer

University of Hamburg and University of California, Berkeley

1. Introduction. Let ξ be a one-dimensional random variable distributed according to $N(a, \sigma^2)$ (that means a normal distribution with mean value a and variance σ^2) and $f(x)$ a measurable function defined on $-\infty < x < \infty$. Suppose that

$$((2\pi)^{\frac{1}{2}}\sigma)^{-1}\int_{-\infty}^{+\infty} f(x)e^{-(x-a)^2/2\sigma^2}\,dx = \omega(a, \sigma^2)$$

exists for all real numbers a and all $\sigma^2 > 0$ as a Lebesgue integral. Let λ be a fixed positive number and η a one-dimensional random variable with distribution $N(a, \sigma^2\lambda^2)$. Let ζ be a random variable such that z/σ^2 has a Pearson-Helmert distribution with ν degrees of freedom. Further, we suppose that η and ζ are independent. The question is whether or not there are unbiased estimates $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ where $-\infty < a < \infty$, and where $\sigma^2 > 0$. In a paper of Neyman and Scott [1] it is proved that there is always an unbiased estimate $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ for the class of entire functions $f(x)$ which satisfy the conditions

$$(1) \qquad \frac{1}{n}(|f^{(2n)}(0)|)^{1/n} = o(1), \qquad \frac{1}{n}(|f^{(2n+1)}(0)|^{1/n} = o(1)$$

and which take real values on the real line. Condition (1) can be expressed in the following way: $f(x)$ is an entire function of order 2 and type zero or of any smaller order. Let us recall that an entire function $f(x)$ is of order k and type $\alpha \geqq 0$ if $|f(re^{i\varphi})| = O(\exp\{(\alpha + \epsilon)r^k\})$ for every $\epsilon > 0$ but for no $\epsilon < 0$ if $r \geqq r(\epsilon)$ and $0 \leqq \varphi < 2\pi$.

In addition, the following problem is raised in the paper just mentioned: Let $f(x)$ be a measurable function defined on the whole real line such that

(i) $\int_{-\infty}^{+\infty} |f(x)|e^{-\epsilon x^2}\,dx$ converges for all $\epsilon > 0$.

Is there always an unbiased estimate $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ if $f(x)$ satisfies Condition (i)? In this paper it will be shown that there is for each $f(x)$ satisfying Condition (i) an unbiased estimate $H(\eta, \zeta)$ for $\omega(a, \sigma^2)$ if λ is any positive number in the interval $0 < \lambda \leqq 1$. However, if λ is allowed to be >1, then the unbiased estimate $H(\eta, \zeta)$ need not exist. Also, it will be shown that the theorem of Neyman and Scott mentioned above is, in a certain sense, best.

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