PROBABILITY CONTENT OF REGIONS UNDER SPHERICAL NORMAL DISTRIBUTIONS, I[1]

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1. Introduction. The primary purpose of this series of papers is to attempt to lay the groundwork for a relatively well-rounded theory of the spherical normal distribution. Many distributional problems in mathematical statistics may be regarded as particular instances of one general problem, the determination of the probability content of geometrically well-defined regions in Euclidean N-space when the underlying distribution is centered spherical normal and has unit variance in any direction. Specifically then, we require for a definite region R

$$P(R) = (2\pi)^{-\frac{1}{2}N} \int_{\mathbf{x} \varepsilon R} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \, d\mathbf{x}, \tag{1.1}$$

in which $\mathbf{x}' = (x_1, \cdots, x_N)$. The class of problems represented by (1.1) is a very broad one and the literature on it is correspondingly quite enormous and well-diffused. In fact, all the distributional problems which occur in the theory of sampling from multivariate normal populations may in principle be brought under our general heading. Thus, let $\mathbf{y}_i$, $i = 1, 2, \cdots, n$, denote n mutually independent k-dimensional vectors each of which is governed by the elementary probability density

$$p(\mathbf{y}) = (2\pi)^{-\frac{1}{2}k} | \mathbf{V} |^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{y}'\mathbf{V}^{-1}\mathbf{y}}. \tag{1.2}$$

The joint probability density function for the n vectors is $\prod_1^n p(\mathbf{y}_i)$ and integrals of the form

$$\begin{aligned} &(2\pi)^{-\frac{1}{2}nk} | \mathbf{V} |^{-\frac{1}{2}n} \int_{\mathbf{z} \varepsilon T} \exp \left(- \tfrac{1}{2} \sum_1^n \mathbf{y}_i' \mathbf{V}^{-1} \mathbf{y}_i \right) \prod_1^n d\mathbf{y}_i \\ &\qquad = (2\pi)^{-\frac{1}{2}N} | \mathbf{W} |^{-\frac{1}{2}} \int_{\mathbf{z} \varepsilon T} \exp (- \tfrac{1}{2}\mathbf{z}' \mathbf{W}^{-1} \mathbf{z}) \, d\mathbf{z}, \end{aligned} \tag{1.3}$$

where $\mathbf{z}$ is a partitioned vector, $\mathbf{W}$ is a partitioned matrix,

$$\mathbf{z} = \begin{bmatrix} \mathbf{y}_1 \\ \cdots \\ \mathbf{y}_2 \\ \cdots \\ \vdots \\ \cdots \\ \mathbf{y}_n \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{V} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots & & \cdots & & \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{V} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots & & \cdots & & \cdots & & \cdots \\ \vdots & & \vdots & & \ddots & & \vdots \\ \cdots & & \cdots & & \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & \mathbf{V} \end{bmatrix}, \tag{1.4}$$

Received February 13, 1959; revised March 5, 1960.

[1] This research was sponsored in part by the Office of Naval Research under Contract Number Nonr-266 (33), Project Number NR 042-034. Reproduction in whole or part is permitted for any purpose of the United States Government.