ON THE ESTIMATION OF THE SPECTRUM OF A STATIONARY STOCHASTIC PROCESS

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1. Introduction. Recently many authors have been interested in the problem of estimating the spectral density function of a weakly stationary process. Under assumptions of linearity of the process and existence of derivatives of the spectral density, U. Grenander and M. Rosenblatt [1] have investigated the asymptotic behaviour of various estimates. E. Parzen [2] has investigated the asymptotic behaviour of different types of errors of the estimates under assumptions of fourth order stationarity and exponential or algebraic decrease of the covariance sequence.

In this paper, the problem of estimating the spectral distribution as well as the spectral density (if it exists) of a weakly stationary process is solved under the sole assumption that the sample covariances converge almost surely and in mean to the true covariances. The relevance of Bochner's work on Fourier analysis [3], in obtaining more exact expressions for the bias of estimates, is pointed out. The existence of estimates which converge uniformly strongly to the spectral density of the process is proved under the assumption that the density has an absolutely convergent Fourier series. It should be added that only questions of consistency are discussed here and, no attempt is made to derive the asymptotic distribution of the estimates.

2. Estimates of the Spectral Distribution Function.

Definitions: We suppose that $x_1, x_2, \cdots x_N$ are observations at N consecutive time points on a discrete weakly stationary stochastic process

$$\{x_t\}(t = \cdots, -1, 0, 1, \cdots),$$

with the well-known spectral representation (cf. [1])

$$(2.1)\quad x_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dZ(\lambda); \qquad Ex_t = 0; \qquad \rho_\nu = \rho_{-\nu} = Ex_t\, x_{t+\nu} = \int_{-\pi}^{\pi} e^{i\nu\lambda}\, dF(\lambda),$$

where $Z(\lambda)$ is an orthogonal stochastic set function (cf. [1]) and $F(\lambda)$ is a monotonic right continuous function in $[-\pi, \pi]$. It is easily seen that

$$(2.2)\qquad \hat{\rho}_\nu = \hat{\rho}_{-\nu} = (x_1 x_{1+|\nu|} + \cdots + x_{N-|\nu|} x_N)/(N - |\nu|)$$

is an unbiased estimate of ρ_ν. We shall consider the following estimate of the spectral distribution:

$$(2.3)\qquad \hat{F}_N(\lambda) = 1/2\pi \sum_{k=-R(N)}^{+R(N)} a_{k,N} \cdot (\hat{\rho}_k/ik) e^{ik\lambda},$$

Received July 20, 1959; revised January 19, 1960.

[1] $\to$ implies the usual weak convergence of distributions.

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