THEOREMS CONCERNING EISENHART'S MODEL II[1]

By Franklin A. Graybill and Robert A. Hultquist

Colorado State University and Oklahoma State University

1. Introduction. Eisenhart's Model II has been discussed in many papers [1], [2], [3], and, since it has become quite important as a statistical model it seems worthwhile to investigate it in some generality. The purposes of this paper are (1) to study the covariance matrix of certain cases, (2) to give some theorems concerning minimal sufficient statistics, (3) to give some theorems concerning best quadratic unbiased estimation, (4) to give some theorems concerning analysis of variance.

2. Notation, Definitions, and Assumptions. In this paper we consider Eisenhart's Model II [4] which can be described as follows. An $n \times 1$ vector of observation $\mathbf{Y}$ is assumed to be a linear sum of $k + 2$ quantities,

$$\mathbf{Y} = \sum_{i=0}^{k+1} \mathbf{X}_i \boldsymbol{\beta}_i, \tag{1}$$

where $\boldsymbol{\beta}_0 = \mu$ is a fixed unknown constant, $\boldsymbol{\beta}_i$ $(i = 1, \cdots k)$ is a vector of p_i random variables, $\boldsymbol{\beta}_{k+1} = \mathbf{e}$ is an $n \times 1$ vector of random errors, $\mathbf{X}_0 = \mathbf{j}$ is an $n \times 1$ vector of 1's, $\mathbf{X}_i$ $(i = 1, \cdots k)$ is a matrix of known constants, and $\mathbf{X}_{k+1} = \mathbf{I}$ is the identity matrix.

Throughout this paper we assume all random variables in and between the vectors $\boldsymbol{\beta}_i$ are independent. $\mathbf{0}$ will denote the null matrix and $\boldsymbol{\beta}_i$ will be distributed with mean $\mathbf{0}$ and covariance matrix $\sigma_i^2 \mathbf{I}$. The covariance matrix of the vector $\mathbf{Y}$ will be denoted by $\mathbf{V}$ and $\mathbf{W}$ will denote $E(\mathbf{YY}')$. $\mathbf{Y}'$ denotes the transpose of $\mathbf{Y}$. Throughout the paper E is the operator denoting the expected value of what follows. $\mathbf{A}_i$ will denote $\mathbf{X}_i \mathbf{X}_i'$ and $\mathbf{A}_i$ $(i = 0, 1, \cdots k + 1)$ will be assumed linearly independent. $\mathbf{J}$ will denote the matrix $\mathbf{jj}'$.

Some of the following assumptions are made in certain sections of this paper.

(i) $\boldsymbol{\beta}_i$ $(i = 1, \cdots k + 1)$ have multivariate normal densities.

(ii) Finite third (fourth) moments exist for all random variables and third (fourth) moments are equal for all variables in a given vector $\boldsymbol{\beta}_i$.

(iii) $\mathbf{A}_i$ and $\mathbf{A}_j$ commute $(i, j = 0, 1, \cdots k + 1)$.

(iv) The matrix $\mathbf{X}_i$ is such that $\mathbf{j}_n' \mathbf{X}_i = r_i \mathbf{j}_{p_i}'$ and $\mathbf{X}_i \mathbf{j}_{p_i}' = \mathbf{j}_n$, where r_i is a positive integer and the subscripts n and p_i are the dimensions of the vectors $\mathbf{j}$.

Many of the commonly used models satisfy most of the above assumptions. For instance, the regression model is included in our discussion when assumptions (iii) and (iv) are deleted. The experimental design models with *equal* numbers in the subclasses satisfy the assumptions. These include the n way cross classifi-

Received April 27, 1959; revised June 27, 1960.

[1] Research sponsored by the National Science Foundation, Grant No. N.S.F. G-3970.