AN ASYMPTOTIC FORMULA FOR THE DIFFERENCES OF THE POWERS AT ZERO

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1. Introduction. In this paper saddlepoint approximations will be obtained for the Stirling numbers. Most of the discussion will be concerned with Stirling numbers of the second kind, which are essentially the same thing as the differences of the powers of the integers at zero, $\Delta^t 0^r$. The work is a direct application of a saddlepoint theorem, Theorem 6.1 of Good [4], which was itself an extension of a result given by Daniels [2]. This theorem enables us to approximate the coefficients in a power of a power series in one variable having non-negative real coefficients.

2. Differences of Powers at Zero. If the sequence 0^r, 1^r, 2^r, is differenced t times, the result for argument 0 is commonly denoted by $\Delta^t 0^r$. For example, $\Delta^2 0^r = 2^r - 2 \cdot 1^r + 0^r$, and generally

$$\Delta^t 0^r = t^r \left\{ 1 - t\left(\frac{t-1}{t}\right)^r + \binom{t}{2}\left(\frac{t-2}{t}\right)^r - \cdots \right\}. \tag{1}$$

This formula is an immediate consequence of the binomial theorem, if Δ is written in the form $E - 1$, where E is the "suffix-raising operator". See, for example, Riordan [6], p. 13.

The differences of the powers at zero are essentially the same thing as the Stirling numbers of the second kind, since $\Delta^t 0^r = t! S(r, t)$. (The notation is that used, for example, by Riordan [6], p. 91.) A table of $\Delta^t 0^r$ for $r \leqq 25$ was presented by Stevens [7], and republished by Fisher and Yates [3], Table XXII. When a power, x^t, is expressed as a linear combination of factorial powers, $S(r, t)$ is the coefficient of $x^{(r)} = x(x-1) \cdots (x - r + 1)$.

When r objects are thrown equiprobably into N cells, the probability that precisely t are occupied is

$$\frac{1}{N^r} \cdot \frac{N!}{(N-t)!\,t!} \Delta^t 0^r. \tag{1a}$$

In a sense this is true even if $t > r$, since $\Delta^t 0^r$ then vanishes. The question of calculating numerical values arises only if $r \geqq t$. In order to emphasise this fact I shall write $r = t + n$, where $n \geqq 0$.

The problem of testing for equiprobability of a multinomial distribution arises in various practical problems, some of which are listed in Good [4], p. 862. Various tests are discussed in this reference, together with conditions under which they are appropriate. Stevens [7] gives two examples, one from agriculture and one from genetics, in which it seems appropriate to use the number of empty

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