RECURRENT GAMES AND THE PETERSBURG PARADOX[1]

By Herbert Robbins

Columbia University

1. Introduction. A *recurrent game* $\mathcal{G}$ is defined by a sequence of trials of a certain, recurrent event $\mathcal{E}$ [1, pp. 238–242]. Let $X_1, X_2, \cdots$ be the sequence of recurrence times of $\mathcal{E}$, $S_n = X_1 + \cdots + X_n$ being the total number of trials up to and including the nth occurrence of $\mathcal{E}$. The X_n are independent random variables with positive integer values and a common distribution:

$$(1) \qquad \begin{aligned} &p_i = P[X_n = i] \qquad (i, n = 1, 2, \cdots), \\ &p_i \geqq 0, \qquad \sum_1^\infty p_i = 1. \end{aligned}$$

We assume that at each occurrence of $\mathcal{E}$ the player receives a *reward* which is a function of the number of trials since the previous occurrence of $\mathcal{E}$; thus at the kth occurrence of $\mathcal{E}$ the player receives the reward c_{X_k}, where $\{c_i\}$ is a given sequence of constants. The player also pays a *fee* f_k on the kth occurrence of $\mathcal{E}$, where $\{f_i\}$ is another given sequence of constants. On any trial on which $\mathcal{E}$ does not occur no money changes hands. With these rules the game $\mathcal{G}$ is determined by the three sequences of constants

$$(2) \qquad \mathcal{G} = \{p_i, c_i, f_i\}.$$

Let

$$(3) \qquad \begin{aligned} V_n &= \text{amount received by player at the } n\text{th trial} \\ &= \begin{cases} c_{X_k} \text{ if for some } k, \ S_k = n \\ 0 \text{ otherwise,} \end{cases} \\ W_n &= \text{amount paid by player at the } n\text{th trial} \\ &= \begin{cases} f_k \text{ if for some } k, \ S_k = n \\ 0 \text{ otherwise,} \end{cases} \end{aligned}$$

and let

$$(4) \qquad \begin{aligned} T_n &= \text{total amount received by player during the first } n \text{ trials} \\ &= V_1 + \cdots + V_n, \\ U_n &= \text{total amount paid by player during the first } n \text{ trials} \\ &= W_1 + \cdots + W_n. \end{aligned}$$

Received June 9, 1960; revised October 4, 1960.

[1] This research was sponsored in part by the Office of Naval Research under Contract Number Nonr-266(59), Project Number 042-205. Reproduction in whole or in part is permitted for any purpose of the United States Government.