ON THE CODING THEOREM FOR THE NOISELESS CHANNEL[1]

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1. Introduction. The purpose of this paper is to examine the coding theorem for a noiseless channel from a point of view different from the usual one. The idea is to take the base s expansion of a point in the unit interval as a realization of the stochastic process to be coded, and then to relate the compression a given coding achieves to known properties of the unit interval, properties connected with Hausdorff dimension and the Shannon-McMillan theorem. This leads to results which in certain ways are sharper than the ones previously obtained.

Let $\Omega = (0, 1]$ and let $\mathcal{B}$ consist of the Borel subsets of Ω. With each ω we associate its nonterminating base s expansion: $\omega = \sum_{n=1}^{\infty} x_n(\omega)/s^n$, where $x_n(\omega) = 0, 1, \cdots, s - 1$. Then each x_n is a measurable function on Ω. If μ is a probability measure on $\mathcal{B}$ then $\{x_1, x_2, \cdots\}$ becomes a stochastic process. Moreover, any stochastic process with state space (or alphabet)

$$\sigma = \{0, 1, \cdots, s - 1\}$$

can be represented in this form, provided it is atomless. More precisely, let $\{p(a_1, \cdots, a_n)\}$ be any consistent set of finite-dimensional distributions with the property that

$$\lim_{n\to\infty} p(a_1, \cdots, a_n) = 0$$

for any sequence $(a_1, a_2, \cdots)$ of elements of σ. Then there exists a measure μ on $\mathcal{B}$ such that

$$\mu\{\omega : x_k(\omega) = a_k, k = 1, \cdots, n\} = p(a_1, \cdots, a_n).$$

Clearly μ will be atomless, or continuous. We will be concerned with such atomless measures μ under which the process $\{x_n\}$ is stationary and ergodic, that is, with measures μ such that if T is defined by $T\omega = [s\omega]$ then T preserves μ and is ergodic under μ. This representation of a process has been used for other purposes by Harris [7].

For the purposes of this paper a *code is a continuous, nondecreasing function* ϕ *on* [0, 1] *with* $\phi(0) = 0$ *and* $\phi(1) = 1$. With each ω we associate the nonterminating base s expansion of $\phi(\omega)$: $\phi(\omega) = \sum_{n=1}^{\infty} y_n(\omega)/s^n$, where

$$y_n(\omega) = 0, 1, \cdots, s - 1.$$

Thus ϕ is a scheme for associating with each sequence $x = (x_1, x_2, \cdots)$ of symbols from σ another such sequence $y = (y_1, y_2, \cdots)$. (For simplicity of

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