A NOTE ON SIMPLE BINOMIAL SAMPLING PLANS

By B. Brainerd and T. V. Narayana[1]

University of Toronto and University of Alberta

Introduction. This note gives two equivalent characterizations of simple sampling plans (s.p.'s) of size n, both of which prove the following Theorem: *The number of simple sampling plans of size n is* $n^{-1}\binom{3n}{n-1}$. The definitions and notations used will be those of M. H. DeGroot [1].

Proof of the Theorem. We indicate only the main steps in the proof, as the details are straightforward and can be filled in by reference to [1].

1. A simple s.p. of size n is characterized by the set C of its continuation points in the lattice quadrant.

2. A set C of lattice points in the quadrant is the set of continuation points of a simple s.p. S of size n if and only if

(i) the intersection C_k of C with each diagonal

$$A_k = \{x + y = k; x \geqq 0, y \geqq 0\}$$

is connected.

(ii) C_k is non-empty if and only if $k < n$.

(iii) No point of C_{k+1} is to the left of the leftmost point of C_k or below the lowest point of C_k. (If A, B are any two points in the lattice plane, A is to the left of B if and only if the x coordinate of A is less than that of B and A is below B if and only if the y coordinate of A is less than that of B).

3. Each non-empty C_k is characterized by how far southeast, t_k, its top is from $(0, k)$ and how far northwest, b_k, its bottom is from $(k, 0)$. t_k, b_k are non-negative integers.

4. The only restrictions on $\{t_k, b_k\}$ of a simple s.p. of size n are

$$t_k + b_k \leqq k, \qquad k = 0, 1, \cdots, n-1,$$
$$0 \leqq t_k \leqq t_{k+1}, \quad 0 \leqq b_k \leqq b_{k+1}, \qquad k = 0, 1, \cdots, n-2.$$

5. The number of different solutions of the above set of inequalities is the number of different simple s.p.'s of size n.

The combinatorial problem posed in 4, 5 may be solved thus. (A more general treatment of such problems is contained in [2].)

If $(x, y)_n$ denotes the number of simple s.p.'s of size n with $t_{n-1} = x$ and $b_{n-1} = y$, then plainly

$$(1) \qquad (x, y)_n = \sum_{a=0}^{x} \sum_{b=0}^{y} (a, b)_{n-1} \qquad \text{for } x + y < n$$
$$= 0 \qquad \text{for } x + y > n.$$

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