A COMBINATORIAL LEMMA FOR COMPLEX NUMBERS[1]

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Although combinatorial lemmas have been used quite successfully in analyzing sums of random variables [1, 2], to the best of our knowledge these considerations have been restricted to the case of real numbers and real variables. It is our purpose in this note to show by a simple example that combinatorial lemmas for complex numbers can also be given and applied to analyzing random walks in the plane.

1. Random walks in the plane. Let $\{Z_k\}$ be a sequence of independent, identically distributed complex-valued random variables. Let $S_0 = 0$, and let $S_n = Z_1 + \cdots + Z_n$, $n \geqq 1$. We call $S_0, S_1, \cdots, S_n, \cdots$ a *random walk in the plane*. The combinatorial lemmas given below are concerned with the *convex hull* of the random walk. Specifically, every walk $S_0, \cdots, S_n$ ($n + 1$ points in the plane) determines a smallest closed, convex set containing these points. The boundary of this set is called the (convex) hull[2] of $S_0, \cdots, S_n$. Later, we will be concerned with three properties of the hull of a walk. We list these properties in the form of variables for later reference.

K_n : the number of variables $Z_1, \cdots, Z_n$ which are line segments in the hull of $S_0, \cdots, S_n$,

(1) H_n : the number of line segments (sides) in the hull of $S_0, \cdots, S_n$,

L_n : the length of the hull of $S_0, \cdots, S_n$.

2. Combinatorics. Let $z_1, z_2, \cdots, z_n$ be a set of n complex numbers and let $s_k = z_1 + \cdots + z_k$. If σ: $i_1, i_2, \cdots, i_n$ is any permutation of $1, 2, \cdots, n$, we let $s_k(\sigma) = z_{i_1} + \cdots + z_{i_k}$. The notation $\vec{z}_A$ will represent the sum of the vectors in a subset A of $z_1, \cdots, z_n$ while z_A will denote the (non-directed) line segment corresponding to $\vec{z}_A$. We need an important definition which seems to be the natural analogue of "rational independence" for real numbers.

Definition. Let $z_1, \cdots, z_n$ be complex numbers with partial sums $s_0, \cdots, s_n$. We say the vectors $z_1, \cdots, z_n$ are *skew* if z_A is parallel to z_B only when $A = B$.

Every vector z in the plane, when extended along its length, determines two half-planes which we call the right and left half-planes of z, respectively. We include the line itself in both of the half-planes.

Lemma 1. *Let $z_1, \cdots, z_n$ be skew vectors with sum z. Then, there exists exactly*

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[2] Some authors call this the boundary of the hull.