ON A COINCIDENCE PROBLEM CONCERNING PARTICLE COUNTERS

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1. Introduction. A general model of particle counting will be considered. Suppose that particles arrive at a counting device at the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$, where the inter-arrival times $\tau_n - \tau_{n-1}$ ($n = 1, 2, \cdots; \tau_0 = 0$) are identically distributed, independent, positive random variables with distribution function $\mathbf{P}\{\tau_n - \tau_{n-1} \leqq x\} = F(x)$, $n = 1, 2, \cdots$. Suppose that each particle, independently of the others, on its arrival gives rise to an impulse either with probability $p(0 < p \leqq 1)$ if at this instant there is at least one impulse present or with probability 1 if there is no impulse present. Let $q = 1 - p$. Denote by χ_n the duration of the impulse (if any) starting at τ_n. It is supposed that $\{\chi_n\}$ is a sequence of identically distributed, independent, positive random variables with distribution function

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{if} \quad x \geqq 0, \\ 0 & \text{if} \quad x < 0, \end{cases} \tag{1}$$

and independent of $\{\tau_n\}$ and the events of realizations of the impulses.

Denote by $\eta(t)$ the number of impulses present at the instant t. Always $\eta(0) = 0$. We shall say that the system is in state E_k, $k = 0, 1, 2, \cdots$, at the instant t if $\eta(t) = k$. Write $\mathbf{P}\{\eta(t) = k\} = P_k(t)$. Furthermore, denote by $\nu_t^{(k)}$ the number of transitions $E_k \to E_{k+1}$ ($k + 1$ -fold coincidences, $k = 0, 1, 2, \cdots$) occurring in the time interval $(0, t]$. Write $\mathbf{E}\{\nu_t^{(k)}\} = M_k(t)$.

The stochastic behavior of the process $\{\eta(t); 0 \leqq t < \infty\}$ is characterized by two parameters, p and μ, and the distribution function $F(x)$. Throughout this paper μ will always be fixed and only p and $F(x)$ will vary. For the sake of brevity we shall say that the process $\{\eta(t); 0 \leqq t < \infty\}$ is of type $[F(x), p]$.

In what follows we shall give a method to determine the distributions of the random variables $\eta(t)$ and $\nu_t^{(k)}$ for finite t and the corresponding asymptotic distributions as $t \to \infty$. The above mentioned problems for process of type $[F(x), 1]$ were solved earlier by the author [13], [14]. The present model of particle counting in the particular case of Poisson input was introduced by G. E. Albert and L. Nelson [1] and generalizations have been given by the author [10], [12], R. Pyke [7], and W. L. Smith [9].

2. The structure of the process, $\{\eta(t)\}$. The stochastic behavior of the process of type $[F(x), 1]$ is already known [14]. Now we shall show that the investigation of the process of type $[F(x), p]$ can be reduced to that of the process of type

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