ON MARKOV CHAIN POTENTIALS[1]

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1. Introduction. In [3] we developed a theory of potentials for denumerable Markov Chains. The purpose of this note is to supplement these results in two ways: We will show for an important special class of Markov chains that they are normal (i.e., that the potential operators exist), and we will generalize certain results due to Spitzer [5].

While our previous paper developed a theory both for transient and for recurrent chains, our present note will deal only with the recurrent case. The key definitions, notations, and theorems for this type of chain will be summarized below. Parenthetical references to theorems will always refer to [3], Section 3.

We consider both measures (row vectors) and functions (column vectors); the former are denoted by Greek letters, the latter by ordinary lower case letters. The theory for functions is dual to that for measures. One passes from one to the other by replacing a transition matrix $\{P_{ij}\}$ by the "reverse chain" $\{\alpha_j P_{ji}/\alpha_i\}$, where $\alpha > 0$ and $\alpha P = \alpha$.

If the limit $\nu = \lim_n [\mu(I + P + \cdots + P^n)]$ exists, we say that ν is a potential, and μ is its charge. The set of states for which μ_i is non-zero is the support of the charge. If $\mathbf{1}$ is the constant function, and if $\mu\mathbf{1}$ is defined, then $\mu\mathbf{1} = 0$. Dually, one defines potential functions. If the column vector f is a charge of a potential function, and αf is finite, then $\alpha f = 0$.

Let $N_{ij}^{(n)}$ be the mean of the number of times that the process is in state j in the first n steps, starting at i. If $\lim_n [N_{jj}^{(n)} - N_{ij}^{(n)}] = C_{ij} \geqq 0$ exists for all i and j, we say that the chain is normal. Under certain assumptions, if ν exists then $\nu = -\mu C$. A sufficient condition is that μ be a weak charge, i.e., that not only μC is finite but also Cf, where $f_i = \mu_i/\alpha_i$ is the dual charge. (See Theorem 15.) For example, all charges of finite support are weak. The dual operator $G_{ij} = \lim_n [N_{ii}^{(n)} \cdot \alpha_j/\alpha_i - N_{ij}^{(n)}]$ serves a similar role for functions. All ergodic (positive recurrent) chains are normal, and the finiteness of μC suffices to assure the existence of the potential.

Many of our considerations will be relative to a given set of states E. Then B_{ij}^E is the probability of entering E at j, starting at i. ${}^E N_{ij}$ is the mean number of times in j, starting at i, before hitting E—this is taken to be 0 if i or j is in E, and we write ${}^k N_{ij}$ if $E = \{k\}$. By P_{ij}^E we mean the probability that starting from i in E we reenter E at j; P^E is itself a recurrent transition matrix, for the states in E. The submatrix of C consisting of rows and columns in E is denoted by C_E.

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