A CENTRAL LIMIT THEOREM FOR PARTLY DEPENDENT VARIABLES

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1. Introduction and definitions. It is well-known that the Central Limit Theorem can be extended to cases in which the random variables under consideration are not entirely independent. In particular, various theorems have been produced with the purpose of dealing with variables which are dependent only when they are in some sense near to each other. The case of m-dependent variables (see, for instance, [13] and [16]) belongs to this category. Another case of this kind arises when the variables have several indices and are regarded as near to each other when they have at least one index value in common (for a somewhat special instance of this case see [9]). The importance of the latter case is due to the fact that it covers a large class of statistics for which W. Hoeffding [5] suggested the name of U-statistics; however, these statistics are only a special instance of it, as can be seen from the reduced number of degrees of freedom.

The purpose of the present paper is to prove a general form of the Central Limit Theorem for partly dependent variables. Its statement is believed to include, as special cases, all the hitherto published propositions on these lines, to cover most, if not all, the situations which have been treated ad hoc, and to go, in some directions, beyond the previously obtained results. As remarked by Feller [2], limiting distributions of normalized sums of random variables should not depend on the existence of moments; accordingly, no moments are postulated, and indeed the most general form of the Central Limit Theorem for independent random variables [2] is contained in the theorem which follows. The statement of the latter may appear slightly cumbersome but it implies, as corollaries, a variety of simpler propositions which are given in Section 3; on the other hand, its proof, which is a generalization of the argument in [16], and does not reduce the general case to that of independent variables, remains conceptually as simple as it would be if the argument were confined to some of the special cases of partly dependent variables. In order to simplify the language, the whole argument is stated for one-dimensional variables, but there is no difficulty in extending it to multi-dimensional variables; a general expression for the mixed moments given, for instance, in [7] is useful in applying the multivariate form of the Second Limit Theorem (discussed, for instance, in [10], Section 7).

In order to avoid misunderstandings, it should be remembered that pairwise disjoint sets of random variables are called (mutually) *independent* if the joint probability distribution function of their union is the product of the joint probability distribution functions of the various sets. A set of random variables will be called *irreducible* if it cannot be decomposed into two (mutually) independent proper subsets. But the factorization of a joint probability distribution function

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