NOTES

ON THE CHAPMAN-KOLMOGOROV EQUATION[1]

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A partial answer is given to the question of whether every Markov random function comes from a system of transition probabilities satisfying the Chapman-Kolmogorov equation. A given Markov random function determines the transition probabilities up to sets of probability zero and for any choice of the transition probabilities the Chapman-Kolmogorov equation holds up to sets of probability zero. The problem then is one of selecting appropriate versions of the transition probabilities so that the Chapman-Kolmogorov equation holds everywhere. It is shown that such selections exist whenever the time parameter set is countable or whenever the joint distribution of any two of the random variables is absolutely continuous with respect to the product of the marginal distributions. Although the latter condition is always satisfied when the state space is countable, or more generally, when each random variable assumes a countable number of values with probability one, this case, being especially simple, is treated separately. The results are based on exploiting the device of using the marginal distribution when in doubt about what the conditional probability distribution should be.

Let $(X_t, t \varepsilon T)$ be a Markov random function, where T is a set of real numbers with elements denoted by r, s, t, u, v. Let $\mathcal{S}$ be the σ-field of linear Borel sets, and for every t define $P_t(S) = P[X_t \varepsilon S]$, $S \varepsilon \mathcal{S}$. For every s, t, $s < t$, consider the joint probability distribution of X_s, X_t. There exists what we shall call a version of the conditional probability distribution of X_t given X_s or, more concisely, a version of $P(X_t \mid X_s)$, that is, a function P_{st} of x, S, x real, $S \varepsilon \mathcal{S}$, such that $P_{st}(\cdot, S)$ is Borel for every $S \varepsilon \mathcal{S}$, $P_{st}(x, \cdot)$ is a probability distribution on $\mathcal{S}$ for every x, and

$$\int_S P_s(dx)P_{st}(x, S') = P[X_s \varepsilon S, X_t \varepsilon S'], \qquad S, S' \varepsilon \mathcal{S}.$$

The Markov property implies that for $r < s < t$, $P_{rs}*P_{st}$ is a version of $P(X_t \mid X_r)$, where by definition

$$(P_{rs}*P_{st})(x, S) = \int P_{rs}(x, dy)P_{st}(y, S), \qquad \text{all } x, S \varepsilon \mathcal{S},$$

Received September 26, 1960; revised March 27, 1961.

[1] This paper was prepared with the partial support of the Office of Ordnance Research, U. S. Army under Contract No. DA-04-200-ORD-171, Task Order 3.