TWO SIMILAR QUEUES IN PARALLEL

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1. Introduction. Haight [3] has considered a system consisting of two unbounded single server queues, in which a customer, on arrival, joins the shorter queue. In the present paper, we make the simplifying assumption of symmetry between the two queues, an assumption that enables us to use generating functions to study the behavior of the stationary solution.

Thus we assume that the two servers each have an exponential service time distribution with unit mean, and that the arrivals form a Poisson process with mean 2ρ. If an arriving customer finds that both queues have equal length, he joins either with probability $\frac{1}{2}$.

We first prove that, so long as $\rho < 1$, a state of statistical equilibrium is reached. Then the equilibrium equations are converted into an equation for a bivariate generating function, by which this function is given in terms of two univariate generating functions. These two functions are shown to be meromorphic, and the positions of, and residues at, their poles are found. This enables us to express the probabilities as an infinite sum of geometric distributions. It also provides us with approximations valid when ρ is near unity, such as the result that the waiting time distribution of a customer is the same as that for a single queue with traffic intensity ρ^2.

2. Limiting behavior of the system. The first problem to be decided is whether or not the queue will settle down into a stationary state. Under the assumptions that have been made, the lengths of the two queues form a continuous time Markov process, and we first prove a lemma referring to these processes in general, giving a sufficient condition for a valid limiting distribution to exist. This lemma, which is an extension of a theorem of Foster [2] on the discrete time case, is of wide applicability, and it is hoped to publish an account of further extensions elsewhere.

We consider an irreducible Markov process $X(t)$, taking a countable number of values i, and we assume that the limits

$$q_{ij} = \lim_{t\to 0} t^{-1}\{P(X(t) = j \mid X(0) = i) - \delta_{ij}\}$$

exist and satisfy the conservation conditions $\sum_j q_{ij} = 0$.

Lemma 1. *Let $-q_{ii}$ be bounded. Then the limits*

$$p_j = \lim_{t\to\infty} P(X(t) = j \mid X(0) = i)$$

exist and are independent of i. The $\{p_j\}$ form a probability distribution if and only

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