THE TRANSIENT BEHAVIOR OF A SINGLE SERVER QUEUING PROCESS WITH RECURRENT INPUT AND GAMMA SERVICE TIME[1]

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1. Introduction. Let us consider the following queuing process: Customers arrive at a counter at the instants $\tau_0, \tau_1, \cdots, \tau_n, \cdots$ where the interarrival times $\theta_n = \tau_{n+1} - \tau_n$ $(n = 0, 1, \cdots; \tau_0 = 0)$ are identically distributed, mutually independent, positive random variables with distribution function $\mathbf{P}\{\theta_n \leqq x\} = F(x)$. We say that $\{\tau_n\}$ is a *recurrent process*. The customers will be served by a single server. The server is idle if and only if there is no customer waiting at the counter, otherwise the order of the services is irrelevant. The service times are identically distributed, mutually independent random variables with the distribution function

$$H_m(x) = \begin{cases} 1 - \sum_{j=0}^{m-1} e^{-\mu x} \frac{(\mu x)^j}{j!} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{1}$$

and independent of $\{\tau_n\}$.

We are interested in the investigation of the stochastic behavior of the queue size and the busy period of this process. We shall see, however, that if we know the stochastic behavior of the process defined below, then that of the above process can be deduced immediately.

To define the second process let us suppose that customers arrive at a counter in batches of size m at the instants $\tau_0, \tau_1, \cdots, \tau_n, \cdots$, where $\{\tau_n\}$ is the recurrent process defined above. There is a single server. The server is idle if and only if there is no customer waiting at the counter, otherwise the order of the services is irrelevant. The service times are identically distributed, mutually independent random variables with the distribution function

$$\mathrm{H}(x) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{2}$$

and independent of $\{\tau_n\}$.

Denote by $\xi(t)$ the queue size at the instant t, i.e., $\xi(t)$ is the number of customers waiting or being served at the instant t. We say that the system is in state E_k at the instant t if $\xi(t) = k$. Further define $\xi_n = \xi(\tau_n - 0)$, i.e., ξ_n is the queue size immediately before the arrival of the nth batch $(n = 0, 1, \cdots)$.

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