ON A SPECIAL CLASS OF RECURRENT EVENTS

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I. Introduction. Let F be the set of all finite sequences (*words*) in the symbols $x \varepsilon X$. According to W. Feller ([2], Chap. VIII), a recurrent event $\mathcal{E}$ is a pair (A, μ) where A is a subset of F and μ a probability measure fulfilling the conditions recalled below; one says that the event $\mathcal{E} = (A, \mu)$ occurs at the last letter x_{i_n} of a word $f = x_{i_1}x_{i_2} \cdots x_{i_n}$ if and only if f belongs to the set A; we shall call A the *support* of $\mathcal{E}$ and denote by $T(A, \mu)$ the mean recurrence time of the event $\mathcal{E}$.

If the pair (B, μ') defines another recurrent event on F, the pair $(A \cap B, \mu')$ defines also a recurrent event. It results from the general theory of Feller ([2], Chap. VIII) that, when $T(B, \mu')$ is finite, the ratio $\pi = T(B, \mu')/T(A \cap B, \mu')$ is, in a certain sense, the limit of the conditional probability that a random word $f \varepsilon F$ belongs to A when it is known to belong to B. For given arbitrary A, it is in general possible to find infinitely many (B, μ') having finite $T(B, \mu')$ which are such that $\pi = 0$.

The main point of this note is to verify several statements which, together, imply the following property:

Property 1. If the support A *is such that* $T(A \cap B, \mu')$ *is finite for* every recurrent event (B, μ') having *finite* $T(B, \mu')$, *then, for every such* (B, μ'), π^{-1} *is an integer at most equal to a certain finite number* δ^* *which depends only upon* A.

Classical examples of this occurrence are the return to the origin in random walks over a finite group [3] and, in particular, the recurrent event which occurs at the end of every word whose length is an integral multiple of a particular integer.

In Section II, we discuss some properties of a class of recurrent events which we shall call *birecurrent*; in Section III, we verify the statements mentioned above, and in Section IV we describe examples of birecurrent supports.

II. Preliminary remarks. We consider F as the free monoid ([1], Chap. 1) generated by X; the empty word e is the neutral element of F and the product ff' of the words f and f' is the word f'' made up of f followed by f'; $f(f')$ is called a *left* (*right*) *factor* of f''; a word is *proper* if it is different from e.

Feller's condition ([2], Chap. VIII) that the non empty subset A of F is the support of a recurrent event can be expressed as follows: U_r : *if* $a \varepsilon A$ *and* $f \varepsilon F$, *then,* $af \varepsilon A$ *if and only if* $f \varepsilon A$. This condition implies that A is a submonoid of F (i.e., that $e \varepsilon A$ and $A^2 \subset A$). We shall say that A is *birecurrent* if it satisfies U_r and the symmetric condition U_l, U_l : *if* $a \varepsilon A$ *and* $f \varepsilon F$, *then,* $fa \varepsilon A$ *if and only if* $f \varepsilon A$.

It follows immediately that, if $\{A_i\}$ is any collection of supports of recurrent

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