NOTES

PAIRWISE INDEPENDENCE OF JOINTLY DEPENDENT VARIABLES

By Seymour Geisser and Nathan Mantel

National Institute of Mental Health and National Cancer Institute

1. Introduction. It is well known in statistical theory that pairwise independence is necessary but not sufficient for a set of $p \geqq 3$ variables to be mutually independent. The example of a set of three discrete variates that are pairwise independent but not mutually independent that is usually quoted in the statistical literature is due to S. Bernstein (see Cramér [2, page 162] or for a variation, Parzen [4, page 90]). In fact Feller [3, page 117], commenting on this example, states that "Practical examples of pairwise independent events that are not mutually independent apparently do not exist."

Two additional discrete examples of such pairwise independence can readily be presented. For the first, consider the case where p straight line segments are distributed independently and at random on a plane (more appropriately, to avoid marginal effects, these should be great circle segments distributed on the surface of a sphere). The $\frac{1}{2}p(p - 1)$ random variables, X_{ij}, exhibiting pairwise independence equal 1 or 0 according to whether or not segments i and j intersect. In the second example, p balls are distributed independently with equal probability into each of 2 or more urns. The X_{ij}'s in this case equal 1 or 0 according to whether or not balls i and j have been placed in the same urn.

In this note we give a continuous example which may be of both practical interest and pedagogical value.

2. An example. Consider a random sample of n observations from a trivariate non-singular normal distribution with a diagonal variance-covariance matrix. The joint density of the three sample correlation coefficients, say r_{12}, r_{13} and r_{23}, is

$$f(r_{12}, r_{13}, r_{23}) = C(n)(1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23})^{\frac{1}{2}(n-5)} \tag{1}$$

for

$$1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23} > 0 \tag{2}$$

and zero elsewhere.

It is obvious that the three random variables are mutually dependent from positive definiteness and continuity considerations. The pairwise independence of these variables can be shown directly by integrating out in the joint density

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