ENUMERATION OF LINEAR GRAPHS FOR MAPPINGS OF FINITE SETS

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1. Introduction. A finite set of n elements may be mapped onto itself in n^n ways, since each element may be mapped independently on any element. Each mapping is a permutation with unlimited repetition. A linear graph is found for a mapping by drawing a sling if i is mapped on i, a line from i to j if i is mapped on j. Hence each linear graph consists of one or more components (connected parts) and each component has a single cycle (closed path) of length $k = 1, 2, \cdots$, if a sling is regarded as of length 1, and a pair of points connected by two lines a cycle of length 2. Also, all points of the graph are labeled and directedness of the lines has significance for the enumeration only in that, for $k > 2$, the lines in cycle may be directed in two ways. Note that slings and multiple lines between pairs of points usually are banned in graph enumerations.

The study of such mappings may be given a probability setting by assigning a probability for each mapping. In the simplest case, the probability is the same for all, and the mappings are said to be random. Random mappings have been considered in 1953 by N. Metropolis and S. Ulam [7] who raised the question of the expected number of components, answered in 1954 by Martin D. Kruskal [6]. H. Rubin and R. Sitgreaves in 1954 considered other random variables associated with the mappings, including the number of lines in cycles. Both Jay E. Folkert [2] and Bernard Harris [4] have considered the enumeration by number of components, both with and without slings. Finally Leo Katz [5] has enumerated the connected graphs with slings, while Alfréd Rényi [8] has given the corresponding result for the classical case where slings and multiple lines are banned.

In the present paper new and simpler results are obtained for the enumerations both by number of components and by number of lines in cycle of the unrestricted and various restricted graphs.

2. The number of components in unrestricted graphs. The graphs in question are those corresponding to the complete set (n^n) of mappings described above. Let T_{nk} be the number of such graphs with n distinct (labeled) points and k components. Let C_k be the number of such connected graphs with k labeled points. Then if the enumerator by number of parts is

$$T_n(x) = \sum_{k=1}^{n} T_{nk}x^k, \qquad T_0(x) = 1,$$

and if

$$C(y) = \sum_{n=1}^{\infty} C_n \frac{y^n}{n!},$$

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