A LAW OF LARGE NUMBERS FOR THE MAXIMUM IN A STATIONARY GAUSSIAN SEQUENCE[1]

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1. Introduction. Let X_1, X_2, $\cdots$ be sequence of random variables which are unbounded above, and let

$$Z_n = \max (X_1, \cdots, X_n).$$

The law of large numbers (LLN) is said to hold for the sequence $\{Z_n\}$ if there exists a sequence of constants $\{A_n\}$ such that

$$Z_n - A_n \to 0 \qquad \text{in probability.} \tag{1}$$

The necessary and sufficient conditions for the LLN for Z_n in the case where $\{X_n\}$ is a sequence of mutually independent random variables with a common d.f. $F(x)$ were found by B. V. Gnedenko [2]. In particular, he mentioned that the standard normal distribution satisfies the conditions and that (1) holds with

$$A_n = (2 \log n)^{\frac{1}{2}}. \tag{2}$$

The main result of this paper is that if $\{X_n : n \geqq 1\}$ is a stationary Gaussian process with

$$EX_i = 0, \qquad EX_i^2 = 1, \qquad EX_1X_i = r_i,$$

then Z_n satisfies (1) with A_n given by (2), under the condition that $nr_n \to 0$.

Lemma 1 furnishes a condition for a stationary process under which the maximum behaves (in probability) almost as if the underlying random variables were mutually independent. Lemma 2 generalizes a result of G. S. Watson [3] on the tail of a bivariate normal d.f. The results of Lemma 2 are used to show that the given stationary Gaussian process satisfies the conditions of Lemma 1.

2. Gnedenko's conditions. It has been shown by Gnedenko [2] that (1) holds if and only if for every $\epsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} n(1 - F(A_n + \epsilon)) = 0 \\ &\lim_{n\to\infty} n(1 - F(A_n - \epsilon)) = \infty. \end{aligned} \tag{3}$$

This can be seen from the fact that (1) holds if and only if for every $\epsilon > 0$,

$$\begin{aligned} 1 &= \lim_{n\to\infty} P\{A_n - \epsilon < Z_n \leqq A_n + \epsilon\} \\ &= \lim_{n\to\infty} F^n(A_n + \epsilon) - \lim_{n\to\infty} F^n(A_n - \epsilon), \end{aligned} \tag{4}$$

Received April 19, 1961; revised August 21, 1961.

[1] Work done under a grant from the National Science Foundation. This is a section of the author's Ph.D. dissertation written at Columbia University.