GAMES ASSOCIATED WITH A RENEWAL PROCESS

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1. Introduction. Consider a sequence of occurrences of a recurrent event $\mathcal{E}$ for which the intervals, $X_1, X_2, \cdots$, are independent identically distributed non-negative random variables (a renewal process) with common cdf (cumulative distribution function) $F(x)$. Robbins [3] considered games when X_1 is an integer-valued random variable. It seems of interest to extend his results to games when X_1 is not necessarily integer-valued. Thus, for example, $\{X_i\}$ may denote the lifetimes of similar articles, or the times between accidents of automobiles insured by a certain company. We will also consider games associated with *a general renewal process* where $\{X_i\}$ is preceded by another random variable X_0 which is independent of $\{X_i\}$ and may have a different distribution. Since the discrete case has been fully dealt with by Feller [1] and Robbins [3], the emphasis in this paper will be on the continuous case. However, the results will be presented in a general form which will include all such $F(x)$ *which do not have a jump at zero.*

2. Fixed-time games. Let us consider the following game called $\mathcal{G}$. The game starts at $t = 0$ when an event has just occurred. At the kth occurrence of the event $\mathcal{E}$, player A *receives* an amount $c(X_k)$ and *pays* an amount a_k. Here $c(t)$ is a given function which vanishes for $t < 0$, and a_i is a sequence of constants. For example, A may be a buyer of certain articles. The benefit which A derives from the kth article is a function $c(X_k)$ of its lifetime, X_k, and he pays the price a_k for the purchase. An insurance company A pays an amount a_k at the occurrence of kth death (life insurance) or kth accident (automobile or other accident insurance) and receives premiums and interests which are a function of time between such occurrences.

Let

$$
\begin{aligned}
T(t) &= \text{total amount received by } A \text{ in } (0, t];\\
T_i(t) &= \text{total amount received by } A \text{ in } (X_i, X_i + t];\\
U(t) &= \text{total amount paid by } A \text{ in } (0, t].
\end{aligned} \tag{1}
$$

Obviously the $T_i(t)$ have the same distribution as $T(t)$. If $ET(t) = EU(t)$ for all $0 \leqq t \leqq t_0$, we shall say that $\mathcal{G}$ is fair for $[0, t_0]$; if $t_0 = \infty$, we shall say that $\mathcal{G}$ is *fair*.

To avoid triviality we shall assume that $c(t)$ is not a null function. $c(t)$ will be called *realistic* if it is non-negative and finite for finite t. The game $\mathcal{G}$ will be called realizable if, given a realistic $c(t)$, a sequence $\{a_i\}$ of non-negative constants exists for which $\mathcal{G}$ is fair.

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