AN INVARIANCE PRINCIPLE IN RENEWAL THEORY[1]

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1. Introduction. Let $X_1, X_2, \cdots,$ be a sequence of independent, identically distributed positive random variables, and let S_n be their nth partial sum. Define a stochastic process $\{Y_t\}$ by letting

$$Y_t = t - \max\{S_n \mid S_n \leqq t\}; \tag{1.1}$$

$\{Y_t\}$ is Markovian and has stationary transition probabilities. It has been shown by Dynkin [3] (see also [7]) that the random variable Y_t/t has a non-degenerate limiting distribution as $t \to \infty$ if and only if $F(x)$, the common distribution function of the X_i, satisfies

$$1 - F(x) = x^{-\alpha}L(x), \qquad 0 < \alpha < 1, \tag{1.2}$$

where $L(x)$ is a slowly varying function. (That is, $L(cx)/L(x) \to 1$ as $x \to \infty$ for every positive c. These functions were introduced by Karamata in [5].) More recently it has been shown [8] that the same is true of M_t/t, where

$$M_t = \sup_{\tau \leqq t} Y_\tau . \tag{1.3}$$

These facts suggest that perhaps, if (1.2) holds, the *processes* $\{\xi^{-1}Y_{\xi t}\}$ converge as $\xi \to \infty$ to a limiting process in somewhat the manner of Donsker's *invariance principle* [2]. The purpose of the present paper is to investigate this question.

In Section 2, it will be shown that the transition probability function of the Markov process $\{\xi^{-1}Y_{\xi t}\}$ converges to a limit as $\xi \to \infty$; the limit is explicitly obtained. It follows easily that the finite-dimensional distributions of the process also converge to ascertainable limits. In Section 3 it is then shown that the limiting distribution of any suitably continuous path functional exists; in other words, that an invariance principle holds. The principle tool (in addition to the convergence of the finite dimensional distributions) is a general theorem of Skorohod [10] which is easily applied to the present situation.

The next two parts of the paper study the limits, and show that they can be identified with secondary processes derived from some well-known objects. For example, let $\{x_t\}$ be a one-dimensional Wiener (Brownian motion) process with $x_0 = 0$ and define

$$y_t = t - \max\{\tau \leqq t \mid x_\tau = 0\}. \tag{1.4}$$

The process $\{y_t\}$ thus obtained is the limit process to which $\{\xi^{-1}Y_{\xi t}\}$ converges in the case $\alpha = \frac{1}{2}$. This fact can be generalized to include other values of α by con-

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