# AN EXTENSION OF THE ARC SINE LAW[1]

By Simeon M. Berman

*Columbia University*

**1. Introduction.** Let $X_1, X_2, \cdots$ be a sequence of random variables and let

$$S_0 = 0$$
$$S_n = X_1 + \cdots + X_n; \qquad (n \geqq 1)$$

$L_n$ = index at which $S_i$, $i = 0, 1, \cdots, n$ attains for the first time the value max $(S_0, \cdots, S_n)$;

$M_n$ = index at which $S_i$, $i = 0, 1, \cdots, n$ attains for the last time the value min $(S_0, \cdots, S_n)$;

$N_n$ = number of $S_0, S_1, \cdots, S_n$ which are positive.

The arc sine law is a general class of limit theorems dealing with the limiting distributions of $L_n$, $M_n$ and $N_n$. The first results on the arc sine law were obtained under the assumption of the independence of the $\{X_n\}$ [2], [4], [5]. Later results were obtained using only certain symmetry properties but not necessarily independence [1].

The primary result of this paper is a generalization of E. S. Andersen's arc sine law for sequences of "symmetrically dependent" random variables; this generalization is obtained with the aid of a representation theorem of de Finetti [6, p. 365]. According to that theorem, a sequence of exchangeable (symmetrically dependent) random variables can be represented as conditionally independent random variables. The limiting distributions in the case of exchangeability are obtained from those in the case of independence.

**2. Preliminary results.**

Lemma 1. *Let $\{X_n\}$ be a sequence of independent random variables with a common distribution function $F(x)$. Then $\lim_{n\to\infty} P\{S_n = 0\}$ is 0 or 1 depending on whether $F(x)$ is nondegenerate or degenerate at 0.*

Proof. The degenerate case is trivial; the nondegenerate case follows from a theorem of Doeblin and Lévy [3] according to which the distribution of the sum of a large number of independent random variables, whose dispersions are uniformly bounded away from zero, has at most small jumps at its points of discontinuity.

Lemma 2. *Let $\{X_n\}$ be a sequence of independent random variables with a common nondegenerate distribution function $F(x)$, which is symmetric, i.e., $1 - F(x) = F(-x)$ at all points of continuity. Let $K_n$ denote any of the random variables $L_n$, $n - M_n$, or $N_n$; then*

$$\lim_{n\to\infty} P\{K_n \leqq \alpha n^{\frac{1}{2}}\} = (2/\pi) \sin^{-1} \alpha^{\frac{1}{2}}, \qquad 0 \leqq \alpha \leqq 1. \tag{2.1}$$

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