CONDITIONS FOR WISHARTNESS AND INDEPENDENCE OF SECOND DEGREE POLYNOMIALS IN A NORMAL VECTOR

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1. Introduction. We define a matrix, whose elements are second degree polynomials in a normal vector, as $\mathbf{XAX}' + \frac{1}{2}(\mathbf{LX}' + \mathbf{XL}') + \mathbf{C}$, where $\mathbf{L}$ is a matrix with p rows and n columns (denoted as $\mathbf{L}$: $p \times n$), $\mathbf{A}$:$n \times n$ and $\mathbf{C}$:$p \times p$ are symmetric matrices, and the columns of $\mathbf{X}$:$p \times n$ are independent p-variate normals with means as columns of $\boldsymbol{\mu}$:$p \times n$ and covariance matrix $\mathbf{V}$:$p \times p$. In this paper, we establish the necessary and sufficient conditions for Wishartness and independence of such matrices. The results for $\mathbf{C} = \mathbf{0}$, $\mathbf{L} = \mathbf{0}$ have been established in [1, 3] and for $p = 1$ by R. G. Laha [4].

2. Certain lemmas.

Lemma 1. *Let* $\mathbf{A}$:$n \times n$, $\mathbf{B}$:$n \times n$ *be symmetric matrices, and suppose that* $\mathbf{L}$:$p \times n$ *and* $\mathbf{M}$:$p \times n$ *are matrices such that* t = *rank of* $(\mathbf{A}\ \mathbf{L}')$, u = *rank of* $(\mathbf{B}\ \mathbf{M}')$, $\mathbf{AB} = \mathbf{0}$, $\mathbf{LB} = \mathbf{MA} = \mathbf{0}$ *and* $\mathbf{LM}' = \mathbf{0}$. *Then, there exists a semi-orthogonal matrix* $\mathbf{Q}$:$n \times (t + u)$, $(t + u \leqq n)$, *such that* $\mathbf{L} = (\mathbf{T}\ \mathbf{0})\mathbf{Q}'$, $\mathbf{M} = (\mathbf{0}\ \mathbf{U})\mathbf{Q}'$,

$$\mathbf{A} = \mathbf{Q}\begin{pmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{Q}' \quad \textit{and} \quad \mathbf{B} = \mathbf{Q}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\end{pmatrix}\mathbf{Q}',$$

where $\mathbf{E}$:$t \times t$, $\mathbf{F}$:$u \times u$ *are symmetric matrices,* $\mathbf{T}$:$p \times t$, $\mathbf{U}$:$p \times u$ *and the form of the null matrix* $\mathbf{0}$ *is understood by its context.*

Proof. Using the result (A.3.11) of [5], we can write

$$(\mathbf{A}\ \mathbf{L}') = \mathbf{Q}_1\mathbf{T}_1 \quad \text{and} \quad (\mathbf{B}\ \mathbf{M}') = \mathbf{Q}_2\mathbf{T}_2\,, \tag{2.1}$$

where $\mathbf{Q}_1$:$n \times t(t < n)$, $\mathbf{Q}_2$:$n \times u(u < n)$ are semi-orthogonal matrices, $\mathbf{T}_1 = (\mathbf{T}_{11}\ \mathbf{T}_{12})$ and $\mathbf{T}_2 = (\mathbf{T}_{21}\ \mathbf{T}_{22})$ are of ranks t and u respectively, $\mathbf{T}_{11}$:$t \times n$, $\mathbf{T}_{12}$:$t \times p$, $\mathbf{T}_{21}$:$u \times n$ and $\mathbf{T}_{22}$:$u \times p$. Now by the given conditions, we have $\mathbf{T}_1'\mathbf{Q}_1'\mathbf{Q}_2\mathbf{T}_2 = \mathbf{0}$ and so

$$\mathbf{Q}_1'\mathbf{Q}_2 = \mathbf{0}. \tag{2.2}$$

Hence $\mathbf{Q} = (\mathbf{Q}_1\ \mathbf{Q}_2)$ is a semi-orthogonal matrix, and we can find $\mathbf{Q}_3$:$n \times (n - t - u)$ such that $(\mathbf{Q}\ \mathbf{Q}_3)$ is an orthogonal matrix [5, (A.1.7)]. Using these results, we have from (2.1),

$$(\mathbf{Q}_2\ \mathbf{Q}_3)'(\mathbf{A}\ \mathbf{L}') = \mathbf{0} \quad \text{and} \quad (\mathbf{Q}_1\ \mathbf{Q}_3)'(\mathbf{B}\ \mathbf{M}') = \mathbf{0}. \tag{2.3}$$

Moreover, from (2.1) we can write $\mathbf{L} = (\mathbf{T}_{12}'\ \mathbf{0})\mathbf{Q}'$, $\mathbf{M} = (\mathbf{0}\ \mathbf{T}_{22}')\mathbf{Q}'$ and $\mathbf{Q}_1'\mathbf{A}\mathbf{Q}_1 = \mathbf{T}_{11}\mathbf{Q}_1 = \mathbf{E}$ (say), $\mathbf{Q}_2'\mathbf{B}\mathbf{Q}_2 = \mathbf{T}_{21}\mathbf{Q}_2 = \mathbf{F}$ (say) as symmetric matrices. With the help of (2.3), we can write $\mathbf{A}$ and $\mathbf{B}$ as mentioned in Lemma 1.

Lemma 2. *If the columns of* $\mathbf{X}$: $p \times n$ *are independent p-variate normals with*

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