POISSON PROCESSES WITH RANDOM ARRIVAL RATE[1]

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1. Introduction. Let F be a distribution function on $(0, \infty)$. A probability P_F on the integers defined by

$$P_F(n) = (n!)^{-1} \int_0^\infty e^{-\lambda} \lambda^n \, dF, \qquad n \geqq 0,$$

will be called a mixture of Poisson probabilities. Since [1] there is a 1-1 correspondence between P_F and F, any statistical question about F can, in principle, be answered by random sampling on P_F. However, F can be estimated more easily by random sampling on mixtures of laws of Poisson processes (to be defined below). Even then no unbiased estimate for F exists; but the Glivenko-Cantelli Lemma [2], p. 20 does hold for the natural estimate of a continuous F. These two results are proved in Section 3; Section 2 contains some preliminary material.

2. Independent realizations of mixtures. Let γ be a nonempty set, and $B(\gamma)$ a σ-algebra of subsets of γ. Let $\{P_{\lambda'} : \lambda' \varepsilon \Lambda\}$ be a family of probabilities defined on $B(\gamma)$. Take $B(\Lambda)$ to be the smallest σ-algebra of subsets of Λ over which all the functions $\{P_{\lambda'}(E) : E \varepsilon B(\gamma)\}$ are measurable. If μ is any probability on $B(\Lambda)$, define

$$P_\mu(E) = \int_\Lambda P_{\lambda'}(E) \, d\mu : E \varepsilon B(\gamma).$$

The set function P_μ is again a probability on $B(\gamma)$, and is called a mixture of the probabilities $P_{\lambda'}$. If X is any $B(\gamma)$-measurable function, and P any probability on $B(\gamma)$, define $E(X \mid P) = \int_\gamma X \, dP$. Then

Lemma 1. $E(X \mid P_\mu) = \int_\Lambda E(X \mid P_{\lambda'}) \, d\mu$, *in the sense that if either side exists, both do and they are equal.*

Proof. When X is the characteristic function of a measurable set, the lemma is a restatement of the definition. Hence the lemma holds for all simple functions by linearity, for nonnegative functions by a monotone passage to the limit, and finally for general functions by linearity.

The purpose of the next lemma is to describe mixtures on product spaces. Define ([2], pp. 90–91)

$$(\gamma^J, B(\gamma^J), P^J) = \prod_{j=1}^{J} (\gamma, B(\gamma), P)$$

$$(\Lambda^J, B(\Lambda^J), \mu^J) = \prod_{j=1}^{J} (\Lambda, B(\Lambda), \mu),$$

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