NOTES

A FLUCTUATION THEOREM FOR CYCLIC RANDOM VARIABLES[1]

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1. Statement of theorem. We shall call $X_1, \cdots, X_n$ a *cyclic* set of random variables (or simply *cyclic*) if $P(X_1 \leqq t_1, \cdots, X_n \leqq t_n)$ is constant for all n cyclic permutations of the sequence $t_1, \cdots, t_n$. Loosely speaking, the random variables are cyclic if their distribution law is invariant under cyclic permutations. Similarly, the set is called exchangeable (or symmetrically dependent) if their distribution law is invariant under *all* permutations. Exchangeable sets of random variables are cyclic, but the converse is not true. Let

$$S_k = X_1 + \cdots + X_k, \qquad k = 1, \cdots, n,$$

$$M = \max(S_1, \cdots, S_n),$$

$$x^+ = \begin{cases} x & \text{if } x \geqq 0, \\ 0 & \text{if } x \leqq 0, \end{cases}$$

$$x^- = \begin{cases} x & \text{if } x \leqq 0, \\ 0 & \text{if } x \geqq 0. \end{cases}$$

The main purpose of this note is to prove the following.

Theorem 1. *Suppose that $X_1, \cdots, X_n$ is a cyclic set of random variables. Then*

$$E(M^- \mid S_n = s) = s^-/n. \tag{1.1}$$

The proof will be given in Section 2.

Remark. By the conditional expectation in (1.1) is meant a measurable function of s, which, for any measurable set A on the real line satisfies

$$\int_A E(M^- \mid S_n = s)\, dP(S_n \leqq s) = E(M; S_n \text{ in } A),$$

whenever this expectation exists. The assertion of the theorem is then that s^-/n is one possible version of this function. Of course, any other version must agree with s^-/n, except possibly on a linear set of probability zero.

An interesting special case of (1.1) arises where the X_i assign all their mass to $-1, 0, 1, 2, \cdots$. Then under the condition that $S_n = u$, a negative integer, and M is negative, then M *must equal* -1. Hence, in that case, (1.1) says,

$$P(M < 0 \mid S_n = u) = -u/n. \tag{1.2}$$

A version of (1.2) that has been proved before, but only for exchangeable

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