# A REPRESENTATION OF THE SYMMETRIC BIVARIATE CAUCHY DISTRIBUTION

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**1. Introduction.** There are very few distributions, which, like the normal distribution, have intrinsic extensions to the multivariate situation other than the trivial extension requiring the components to be independent. Of these distributions, the stable distributions occupy a unique position. The multivariate stable distributions may be characterized by the requirement that every one-dimensional marginal distribution (that is the distribution of *every linear* combination of the variables) is a stable distribution. A proof that such a requirement characterizes the multivariate normal distribution may be found in Anderson's book [1], pg. 37. This paper is concerned with an investigation of the symmetric multivariate stable distribution with characteristic exponent 1, namely, the symmetric multivariate Cauchy distribution.

Definition. *A random vector* $\mathbf{X}' = (X_1, \cdots, X_r)$ *is said to have a multivariate Cauchy distribution if, and only if, for every real vector* $\mathbf{t}' = (t_1, \cdots, t_r)$, *the random variable* $\mathbf{t}'\mathbf{X} = \sum t_i X_i$ *has a Cauchy distribution. The distribution is said to be symmetric if the mass is distributed symmetrically with respect to some point in r-dimensional space.*

The following simple lemma is the basis for this study. A similar result for arbitrary stable distributions may be found in Lemma 2 of [2], but note that the word symmetric is used there in a different sense.

Lemma 1. *The distribution of a random vector* $\mathbf{X}$ *is multivariate Cauchy if, and only if, the characteristic function of* $\mathbf{X}$ *has the form*

$$\phi_{\mathbf{X}}(\mathbf{t}) = e^{-g(\mathbf{t}) + i\gamma(\mathbf{t})}, \tag{1}$$

*where* $g(\mathbf{t}) \geqq 0$ *and* $\gamma(\mathbf{t})$ *are real functions satisfying the equations*

$$g(a\mathbf{t}) = |a| g(\mathbf{t}) \tag{2}$$

$$\gamma(a\mathbf{t}) = a\gamma(\mathbf{t}) \tag{3}$$

*for every real number a. If the distribution is symmetric with respect to a point* $\boldsymbol{\gamma}$ *in r-dimensional space, then*

$$\gamma(\mathbf{t}) = \boldsymbol{\gamma}'\mathbf{t}. \tag{4}$$

Proof. If a characteristic function has the form (1) where $g$ and $\gamma$ satisfy (2) and (3), then

$$\begin{aligned} \phi_{\mathbf{t}'\mathbf{X}}(a) &= Ee^{ia\mathbf{t}'\mathbf{X}} = \phi_{\mathbf{X}}(a\mathbf{t}) \\ &= e^{-g(a\mathbf{t}) + i\gamma(a\mathbf{t})} \\ &= e^{-|a| g(\mathbf{t}) + ia\gamma(\mathbf{t})} \end{aligned} \tag{5}$$

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