$(Q_2, Q_3, \cdots, Q_m)$, $(Q_3, Q_4, \cdots, Q_m)$, etc., we finally obtain

$$\Pr\{T_{j1} \leqq \beta_1, T_{j2} \leqq \beta_2, \cdots, T_{jm} \leqq \beta_m\} = \prod_{k=1}^{m} \Pr\{T_{jk} \leqq \beta_k\}$$

as was to be shown.

Finally, it should be mentioned that if the waiting times are defined so as *not* to include the service times, that is, as the quantities $T_{jk} - s_{jk}$, the question of mutual independence of these quantities for $k = 1, 2, \cdots, m$ is apparently an open problem.

REFERENCES

[1] Reich, Edgar (1957). Waiting times when queues are in tandem. *Ann. Math. Statist.* **28** 768–773.
[2] Saaty, Thomas L. (1961). *Elements of Queueing Theory.* McGraw-Hill, New York.
[3] Takács, Lajos (1962). *Introduction to the Theory of Queues.* Oxford Univ. Press, New York.

A NOTE ON THE RE-USE OF SAMPLES[1]

By David R. Brillinger[2]

London School of Economics and Political Science

There are situations in statistical estimation in which the basic underlying distribution is invariant under some family of transformations. In this note a theorem similar to the Blackwell-Rao Theorem is proved demonstrating that this additional structure can sometimes be exploited to improve an estimator.

Theorem. *Consider a random variable x, sample space X, σ-algebra $\mathfrak{X}$, probability measure $P(\ )$. Suppose that G is a set of measure-preserving transformations for the measure P,* i.e. *$P(gA) = P(A)$ for all A in $\mathfrak{X}$, g in G. Let $\mu(\ )$ be a measure of total mass* 1, *defined on a σ-algebra $\mathcal{G}$ of subsets of G. Let $\phi(x)$ be an estimator such that $\phi(gx)$ is $\mathcal{G} \times \mathfrak{X}$ measurable.*

(i) If $\phi(x)$ is an unbiased estimator of θ then,

$$\gamma(x) = \int_G \phi(gx)\, d\mu(g)$$

is also an unbiased estimator of θ.

(ii) If $\phi(x)$ takes values in a k-dimensional space and has an associated real-valued, convex, bounded from below loss function $W[\phi(x)]$, such that $W[\phi(gx)]$ is $\mathcal{G} \times \mathfrak{X}$ measurable then, $R_\phi \geqq R_\gamma$ where R is the associated risk function, and in particular the ellipsoid of concentration of γ is everywhere

Received May 15, 1962.

[1] Part of this research was carried out while the author had the support of a Research Training Fellowship of the Social Science Research Council.

[2] Presently at Bell Telephone Laboratories, Murray Hill, N. J. and Princeton University.