NOTES

THE CONVEX HULL OF PLANE BROWNIAN MOTION

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Denote by $Z(t, \omega)$ the Brownian motion in the plane starting at the origin. Let $J(\omega)$ be the least closed convex set containing $Z(t, \omega)$, $0 \leqq t \leqq 1$, let $K(\omega)$ be the boundary of $J(\omega)$, and let $p(s, \omega)$ be the point on $K(\omega)$ at a distance s along $K(\omega)$ in the counterclockwise direction from the intersection of $K(\omega)$ with the positive x-axis. Let $\theta(s, \omega)$ be the angle made by the tangent to $K(\omega)$ at $p(s, \omega)$ and the x-axis when such a tangent exists. Define

$$\alpha(s, \omega) = [\theta(s, \omega) - \theta(0, \omega)]/2\pi$$

for points where $\theta(s, \omega)$ is defined. Elsewhere let $\alpha(s, \omega) = \lim_{t \uparrow s} \alpha(t, \omega)$. As s increases from 0^- to $l(\omega)$, the length of $K(\omega)$, $\alpha(s, \omega)$ increases from 0 to 1. Hence $\mu[E, \omega] = \int_E d\alpha(s, \omega)$ is a completely additive probability measure on Borel sets in $[0, l(\omega)]$. P. Levy [3] introduced $J(\omega)$ and showed $\mu[E, \omega]$ to be singular with respect to Lebesgue measure. The purpose of this paper is to show that this measure can almost always be concentrated on a set $T(\omega)$ which is small in the sense of Hausdorff measures.

Definition. Let $h(t)$ satisfy (A), namely, be a positive monotone continuous function with $h(0) = 0$. Let $h_\rho^*(E)$ be the greatest lower bound of $\sum_{i>0} h(\operatorname{diam} O_i)$ where the greatest lower bound is taken over all sets $\{O_i\}$ of circles with diameter less than ρ covering E. Define $h^*(E)$, the h-measure of E, by $h^*(E) = \lim_{\rho \to 0} h_\rho^*(E)$.

It is known that $h^*(E)$ is an outer measure in the sense of Carathéodory. For a general discussion of the properties of these measures see [2].

Theorem. *Let $h(t)$ satisfy A and*

$$\lim_{t \to 0} h(t) \log 1/t = 0. \qquad (1)$$

For almost all ω there exists a set $T(\omega)$ in $[0, l(\omega)]$ for which $\mu[T(\omega), \omega] = 1$ and $h^*(T(\omega)) = 0$.

Proof. The proof rests on the following result of Baxter [1]. Let $\{X_i\}$, $i = 1, 2, \cdots$ be independent, identically distributed, complex-valued random variables with uniform angular distribution.[2] Let $S_0 = 0$, $S_i = \sum_{k=1}^{k=i} X_k$. If H_m is the

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[2] Baxter did not use the last restriction.