A CHARACTERIZATION OF THE UNIFORM DISTRIBUTION ON A COMPACT TOPOLOGICAL GROUP

By James H. Stapleton[1]

Purdue University and Michigan State University

1. Introduction. The random variables discussed in this paper take their values in a compact commutative topological group with a countable basis, e.g., the reals (mod 1). The major result is Theorem 4.1, a characterization of the uniform distribution on such a space. It is the similarity of this characterization to that of Skitovic [4] of the normal distribution on the real line which makes it of special interest. The uniform distribution on the spaces considered here seem to play the same central role that the normal distribution does on Euclidean spaces. For example, Theorem 2.3 is a central limit theorem for such space.

Many of the results stated without proof may be found in a 1940 paper by Kawada and Ito [1], which considers the case in which the group operation is non-commutative. The results presented in Sections 2 and 3 are easily obtained, some are well known, and proofs are sometimes omitted. A major purpose of these sections is to fully acquaint the reader with the background necessary to Section 4. Section 5 contains counterexamples which show the necessity of some of the hypotheses of the characterization.

2. Some preliminaries. Let Γ be a compact commutative topological group with a countable basis, the group operation being addition with the symbol $\oplus$. Let $\hat{\Gamma}$ be the character group of Γ; $\hat{\Gamma}$ is also a topological group. The compactness of Γ implies the discreteness of $\hat{\Gamma}$. Denote the value of a character $\hat{x}$ at a point $x \varepsilon \Gamma$ by $(x, \hat{x})$. Let the identities of Γ and $\hat{\Gamma}$ be e and $\hat{e}$ respectively. The character group of the cartesian product of groups such as Γ is the cartesian product of the corresponding character groups. The $\sigma-$ field for each space Γ will always be the class of Borel sets. Examples of Γ and the corresponding $\hat{\Gamma}$ are:

(1) I, the reals (mod 1), and $\hat{I}$, the multiplicative group of all functions $\exp(2\pi inx)$, $x \varepsilon I$, $n = 0, \pm 1, \cdots$.

(2) The product spaces $I^{(n)}$, $I^{(\infty)}$ and $\hat{I}^{(n)}$, $\hat{I}^{(\infty)}$.

(3) Λ_k, the integers (mod k) and $\hat{\Lambda}_k$, the functions $\exp(2\pi itx/k)$, $t = 0, \cdots, n-1$.

(4) Q, the n-adic integers and $\hat{Q}$ the functions $\exp(2\pi im \sum_{k=0}^{h-1} a_k n^k/m_k)$, m and h positive integers, $a_0, a_1, \cdots \varepsilon Q$.

Let (Ω, S, P) be a probability space, let ξ be a random variable defined on Ω and taking values in Γ and let $\mu = P\xi^{-1}$ be the distribution of ξ in Γ. If μ is a Haar measure on Γ, then μ is called the uniform distribution and ξ is said to be

Received January 5, 1962.

[1] Now at Michigan State University.