ON A CLASS OF STOCHASTIC PROCESSES[1]

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1. Introduction. We shall introduce our problem by recalling (see, for instance, [1]) the example of a *branching process* $\{X_t\}$ with mean number of descendants per individual per unit time $= \mu > 1$. It is well known that

$$\Pr\,(\lim_{t\to\infty} X_t/\mu^t = Y \text{ exists}) = 1 \tag{1.1}$$

where Y is a random variable. This result implies that

$$\lim_{s\to\infty} \{X_{s+t}/\mu^s\} = \{Y_t\}, \qquad -\infty < t < \infty, \tag{1.2}$$

exists, where the brackets mean that we are considering the *processes* one of whose random variables is indicated, and the limit is in the sense of convergence of finite-dimensional distributions[3]. Of course, (1.1) implies also that

$$\{Y_t\} = \{\mu^t Y\}, \tag{1.3}$$

so that $\{Y_t\}$ is deterministic in the sense that if its state at some time t is given, the entire past and future are uniquely determined.

The problem studied in this paper is as follows: *Which processes can arise as limits in a manner similar to* (1.2)? That is, if for some stochastic process $\{X_t\}$ in Euclidean space there is a positive measurable function $f(s)$ such that

$$\lim_{s\to\infty} \{X_{s+t}/f(s)\} = \{Y_t\}, \qquad -\infty < t < \infty, \tag{1.4}$$

what can be inferred about the process $\{Y_t\}$? This question is very analogous to the one considered in [2]. The point there was to generalize the limiting operation by which the Wiener process derives from simple random walk under contraction of the space and time scales. The class of limiting processes which can be obtained in that way by varying the starting process (random walk) and the rates of contracting the axes was characterized by the "semi-stable" property. Some special classes of processes having this property were then described. The present investigation has a similar motivation; we are generalizing a well-known theorem (above) involving contraction of the space scale and *translation* of the time axis of one process to obtain another in the limit.

An outline of our results is as follows: in Theorem 1 a characterization is obtained for processes $\{Y_t\}$ which can arise in the manner (1.4). This does not, how-

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[3] All limiting statements for processes are to be interpreted in this way; in addition "equality" of two processes means that they have the same finite-dimensional distributions.