THE DISTRIBUTION OF THE DETERMINANT OF A COMPLEX WISHART DISTRIBUTED MATRIX

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Summary. Let $\boldsymbol{\xi}' = (\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_p)$ denote a p-variate zero mean complex Gaussian random variable with nonsingular Hermitian covariance matrix $\Sigma_\xi = E\boldsymbol{\xi}\bar{\boldsymbol{\xi}}' = \|\sigma_{jk}\|$. The *generalized variance* of $\boldsymbol{\xi}$ is $\sigma_\xi^2 \equiv \det(\Sigma_\xi)$. The real and imaginary parts of the complex random variables $\mathbf{Z}_j$ $j = 1, 2, \cdots p$ are taken to have the special covariance structure described in Goodman [1] and [2] so that the Hermitian covariance matrix Σ_ξ then determines the probability structure of the random variable $\boldsymbol{\xi}$. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_s, \cdots, \boldsymbol{\xi}_n$ denote n independent and identically distributed p-variate zero mean complex Gaussian random variables with Hermitian covariance matrix Σ_ξ. The sample Hermitian covariance matrix $\hat{\boldsymbol{\Sigma}}_\xi \equiv (1/n)\sum_{s=1}^n \boldsymbol{\xi}_s\bar{\boldsymbol{\xi}}_s' \equiv \|\hat{\boldsymbol{\sigma}}_{jk}\|$ is then complex Wishart distributed. The *sample generalized variance* of $\boldsymbol{\xi}$ is $\hat{\boldsymbol{\sigma}}_\xi^2 \equiv \det(\hat{\boldsymbol{\Sigma}}_\xi)$. The random variable $(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2$ is distributed as is the product of p independent χ^2 random variables with $2n$, $2(n-1)$, $\cdots$, $2(n-p+1)$ degrees of freedom respectively.

Definition 1.1. Let $\boldsymbol{\xi}' = (\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_p)$ denote a p-variate zero mean complex Gaussian random variable with nonsingular Hermitian covariance matrix $\Sigma_\xi = E\boldsymbol{\xi}\bar{\boldsymbol{\xi}}' = \|\sigma_{jk}\|$. The *generalized variance* of $\boldsymbol{\xi}$ is $\sigma_\xi^2 \equiv \det(\Sigma_\xi)$.

Comment 1.1. Throughout the paper the real and imaginary parts of the complex random variables $\mathbf{Z}_j$, $j = 1, 2, \cdots p$ are taken to have the special covariance structure described in Goodman [1] and [2] so that the Hermitian covariance matrix Σ_ξ then determines the probability structure of the random variable $\boldsymbol{\xi}$.

Definition 1.2. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_s, \cdots, \boldsymbol{\xi}_n$ denote n independent and identically distributed p-variate zero mean complex Gaussian random variables with Hermitian covariance matrix Σ_ξ. The sample Hermitian covariance matrix $\hat{\boldsymbol{\Sigma}}_\xi \equiv (1/n)\sum_{s=1}^n \boldsymbol{\xi}_s\bar{\boldsymbol{\xi}}_s' \equiv \|\hat{\boldsymbol{\sigma}}_{jk}\|$. The *sample generalized variance* of $\boldsymbol{\xi}$ is $\hat{\boldsymbol{\sigma}}_\xi^2 \equiv \det(\hat{\boldsymbol{\Sigma}}_\xi)$.

Theorem 1.1. The random variable $(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2$ is distributed as is the product of p independent χ^2 random variables with $2n$, $2(n-1)$, $\cdots$, $2(n-p+1)$ degrees of freedom respectively.

Proof. The method of proof is as follows: The characteristic function of the random variable $\ln[(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2]$ is computed. The characteristic function of a random variable which is the sum of p independent $\ln \chi^2$ random variables with $2n$, $2(n-1)$, $\cdots$, $2(n-p+1)$ degrees of freedom respectively is computed. The two characteristic functions are compared and seen to be equal. The characteristic function of the random variable $\mathbf{V} = \ln[(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2]$ is

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