A REMARK ON A PAPER OF TRAWINSKI AND DAVID ENTITLED: "SELECTION OF THE BEST TREATMENT IN A PAIRED-COMPARISON EXPERIMENT"[1]

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The present remark is concerned with the asymptotic behavior of the highest score in a paired-comparison experiment, if the number t of treatments is very large. It answers a question implicitly posed in Fig. 1 A of [1]. Besides, a useful inequality for the joint cumulative distribution function of the scores will be derived.

Assume that the treatments $T_i (i = 1, 2, \cdots, t)$ are all equal, with the exception of a single "outlier" T_t, which will be preferred with probability $p > \frac{1}{2}$ when compared with any other treatment.[2] Assume that each pair of treatments is compared exactly once, and declare best the treatment with the highest score a_i, that is, with the highest number of preferences. Let $P_{p,t}$ be the probability of selecting the actually best treatment T_t by this procedure.

What happens if t tends to infinity? Fig. 1 A of [1] seems to indicate that $P_{p,t}$ tends to 1 if p is near 1, and to 0 if p is near $\frac{1}{2}$.

Rather surprisingly, this conjecture is false; actually we have $\lim_{t\to\infty} P_{p,t} = 1$ for all fixed $p > \frac{1}{2}$.

Consider first the case where all treatments are equal (no outlier). Then each score a_i is binomially distributed, and the reduced score $a_i^* = \{a_i - [(t-1)/2]\}/[(t-1)/4]^{\frac{1}{2}}$ is asymptotically normal with mean 0 and variance 1. In particular (see, e.g., W. Feller, *An Introduction to Probability Theory*, 2nd ed. p. 178).

$$(1) \qquad P[a_i^* > x_{t-1}] \sim [(2\pi)^{\frac{1}{2}} x_{t-1}]^{-1} e^{-\frac{1}{2}x_{t-1}^2}$$

provided $t \to \infty$, $x_{t-1} \to \infty$, $x_{t-1}^3/[(t-1)/4]^{\frac{1}{2}} \to 0$. The sign $\sim$ denotes that the ratio of the two sides tends to 1.

In particular, let $\epsilon > 0$ and put

$$(2) \qquad x_{t-1}^{\pm} = [2\log(t-1) - (1 \pm \epsilon)\log\log(t-1)]^{\frac{1}{2}};$$

one obtains

$$(3) \qquad P[a_i^* > x_{t-1}^{\pm}] \sim \log(t-1)^{\pm\epsilon/2}/(4\pi)^{\frac{1}{2}}(t-1).$$

Now replace treatment T_t by an outlier, $p > \frac{1}{2}$. This stochastically decreases the reduced scores a_i^*, $1 \leqq i < t$, by an amount less than $1/[(t-1)/4]^{\frac{1}{2}}$. But a straightforward calculation shows that we may add a term of the order

Received March 19, 1962.

[1] Prepared with the partial support of the National Science Foundation, Grant G-18792.

[2] To avoid confusion with the number $\pi = 3.14 \cdots$, I changed the π of T. and D. into p.