OPTIMUM PROPERTIES AND ADMISSIBILITY OF SEQUENTIAL TESTS[1]

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1. Introduction and summary. In Wald's sequential probability ratio test (SPRT) [7], with stopping bounds B, $A (B < A)$, for testing a simple hypothesis H_1 against a simple alternative H_2, sampling continues until the first time the probability ratio is either less than or equal to B, or greater than or equal to A. In the former case H_1 is accepted, in the latter case H_2 is accepted. This test is known to have a certain optimum property (OP) first conjectured by Wald [6], [7] Section A7, later proved by Wald and Wolfowitz [9], [10] and partly by Arrow, Blackwell and Girshick [1] (see also Wolfowitz [11]). This OP can be expressed in words rather roughly as follows: Among all sequential tests whose error probabilities do not exceed those of the SPRT under consideration, the latter has the smallest expected sample size under both distributions. However, the validity of the OP has been demonstrated only under two conditions. The first condition is that $B \leqq 1 \leqq A$. The second condition is that only sequential tests with finite expected sample sizes are considered. The question arises then whether these conditions are necessary. The purpose of this paper is partly to show that the second condition is not necessary, but the first one is. The superfluousness of the second condition is shown for a rather general cost function in the following form: If a test has OP among all tests with finite expected sample sizes, then it has OP among all tests. If $B \leqq 1 \leqq A$ does not hold, then the SPRT is not admissible, in a sense which will be made precise in Section 2. This will be demonstrated in a manner which at the same time exhibits a test that not only improves upon the SPRT, but which itself possesses OP. However, if only tests are considered which take at least one observation, then there are no restrictions on A and B (other than $B \leqq A$) for a SPRT to have OP. A summary of the main results appeared in [2]. Some of the results were also obtained by Ghosh [3] under slightly less general conditions.

The methods employed in this paper yield some additional results that are of interest. Part of the Wald and Wolfowitz OP proof [9] consists in showing the existence of a loss function that makes a given SPRT Bayes for a given *a priori* distribution of the two hypotheses. We give a simpler existence proof in Section 5, as well as proving a lemma that, although slightly weaker, achieves the same aim. In Section 6 a certain continuity property of the error probabilities of a SPRT as functions of the stopping bounds is shown. If the lower bound is fixed, the error probabilities are left or right continuous functions of the upper stopping bound depending on whether the latter is a stopping point for the probability ratio or not. Similar statements hold for the error probabilities as functions of the lower stopping bound.

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