TESTS AND CONFIDENCE INTERVALS BASED ON THE METRIC d_2[1]

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1. Introduction. In this paper tests and confidence intervals based on a metric similar to that used in the Kolmogorov-Smirnov tests are introduced. While the tests are slightly more difficult computationally, they have somewhat better discriminating power against certain alternatives. The confidence intervals for probabilities of intervals have the advantage over those given by the Kolmogorov-Smirnov statistics, of shorter maximum length for the same sample size. The statistic studied here was investigated by Kuiper in [4].

2. Definition and properties of d_2. Let $\mathfrak{D}$ be the class of one-dimensional distribution functions, and for F, G in $\mathfrak{D}$, let

$$d_2(F, G) = \sup_{x_2 > x_1} |[F(x_2) - F(x_1)] - [G(x_2) - G(x_1)]|.$$

Theorem 2.1. *d_2 is a metric, $d_1 \leqq d_2 \leqq 2d_1$, where d_1 is the uniform metric,*

$$d_2(F, G) = \sup_{I \text{ an interval}} |P_F(I) - P_G(I)|,$$

$$d_2(F, G) = \sup_x [F(x) - G(x)] + \sup_x [G(x) - F(x)].$$

The proof is a straightforward consequence of the definition of d_2, and also appears in Brunk, [1].

Definition. For $F_o \, \varepsilon \, \mathfrak{D}$, e, $k \, \varepsilon \, [0, 1]$, let

$$C_{e,k,F_o} = \{F \, \varepsilon \, \mathfrak{D}: F_o(x) + e - k \leqq F(x) \leqq F_o(x) + e, \text{ all } x\}.$$

Theorem 2.2. $\bigcup_{e \varepsilon [0,k]} C_{e,k,F_o} = \{F \, \varepsilon \, \mathfrak{D}: d_2(F_o, F) \leqq k\}$.

Proof. Suppose $G \, \varepsilon \, C_{e',k,F_o}$ for some $e' \, \varepsilon \, [0, k]$. For $I = (x_1, x_2]$ the maximum probability that G can assign to I is $[F_o(x_2) + e') - [F_o(x_1) - e' - k] = F_o(x_2) - F_o(x_1) + k$, while similarly the minimum probability G can assign to I is $F_o(x_2) - F_o(x_1) - k$. Hence

$$P_{F_o}(I) - k \leqq P_G(I) \leqq P_{F_o}(I) + k.$$

This holds for each interval I, and since it is true for each $e' \, \varepsilon \, [0, k]$, we have

$$\bigcup_{e \varepsilon [0,k]} C_{e,k,F_o} \subset \{F \, \varepsilon \, \mathfrak{D}: d_2(F_o, F) \leqq k\}.$$

Now suppose $G \, \varepsilon \, \{F \, \varepsilon \, \mathfrak{D}: d_2(F_o, F) \leqq k\}$. Let $\sup_x [G(x) - F_o(x)] = S \leqq k$, $\sup_x [F_o(x) - G(x)] = t \leqq k - S$. Then clearly

$$G \, \varepsilon \, C_{S,S+t,F_o} \subset C_{S,k,F_o} \subset \bigcup_{e \varepsilon [0,k]} C_{e,k,F_o}.$$

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