A MARKOV PROCESS ON BINARY NUMBERS[1]

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1. Introduction. This paper is a study of a certain discrete parameter Markov process in the interval of real numbers (0, 1]. The process is defined by choosing an arbitrary number X_0 in (0, 1], specifying the distribution of a random variable X_1, given X_0, specifying the distribution of a random variable X_2, given X_1, and so on. Let the number X_0 have the binary expansion $X_0 = .\delta_1^{(0)}\delta_2^{(0)} \cdots \delta_n^{(0)} \cdots$, where $\{\delta_n^{(0)}, n = 1, 2 \cdots\}$ is a sequence of zeros and ones. The distribution of a random variable X_1 is determined by the joint distribution of the sequence of digits in its binary expansion, which we now give. Let $\{\delta_n^{(1)}\}$ be a sequence of digits in the binary expansion of X_1. We attribute to the random digit $\delta_1^{(1)}$ a Bernoulli distribution with mean p, that is $\delta_1^{(1)}$ assumes the values 0 and 1 with probabilities $q = 1 - p$ and p, respectively; if $\delta_k^{(0)} = 1$, we set $\delta_{k+1}^{(1)} = 1$, $k = 1, 2 \cdots$; if $\delta_k^{(0)} = 0$, then we give $\delta_{k+1}^{(1)}$ the Bernoulli distribution with mean p, $k = 1, 2, \cdots$. The random digits $\{\delta_n^{(1)}\}$ are assumed to be mutually independent, so that their joint distribution is completed defined. The distribution of X_2, given (the binary expansion of) X_1, is constructed by the same procedure; thus, an entire sequence $\{X_n\}$ is generated. The binary expansion of X_n is written as $.\delta_1^{(n)}\delta_2^{(n)} \cdots$, $n = 1, 2, \cdots$. It is clear that $\{X_n\}$ is a Markov chain with the state space (0, 1]. An initial distribution for the chain is introduced by assigning a distribution to (the digits in the binary expansion of) X_0.

In what follows, a binary expansion which terminates after a finite number of digits 1 will always be written in its non-terminating form, i.e., with an uninterrupted sequence of digits 1 after a finite number of digits 0. For example, the number $.100 \cdots$ will be written as $.011 \cdots$. No ambiguities arise in the transition from one random variable to its successor. If X_n has infinitely many digits 1, so must X_{n+1} by its definition; hence, if X_0 is written with infinitely many digits 1, so is every term in the sequence $\{X_n\}$.

In Section 2, a stationary distribution is constructed and shown to be unique. The absolute distribution of X_n converges only weakly to the stationary distribution, but does not converge over every Borel set of the space (0, 1]. The strong law of large numbers holds in a restricted form for $\{X_n\}$; it is shown that $(n + 1)^{-1}\sum_{j=0}^{n} f(X_j)$ converges with probability 1 to the integral of f with respect to the stationary distribution for every continuous f, but not for every measurable function f.

In Section 3, the states of the chain are classified; Section 4 treats a first passage time problem; Section 5 treats an absorption problem; and Section 6 contains applications to an epidemiological problem.

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