THE FIRST PASSAGE TIME DENSITY FOR HOMOGENEOUS SKIP-FREE WALKS ON THE CONTINUUM

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Introduction. In a recent paper of Keilson [7], the term skip-free was introduced to describe a subclass of random walks $X(t)$ on an ordered set of states x. A walk $X(t)$ may be said to be skip-free in the negative direction for example if, in going from x_2 to $x_1 < x_2$, the walk must pass through all intervening states at least once. A variety of birth and death processes, queuing processes and diffusion processes have this property in one or both directions.

For the description of bounded skip-free processes, employment of the Green's function for the associated unbounded process and a technique of compensation as described in the paper cited, is often advantageous. (A comparison of the compensation technique with Wald's identity is given in Section 6.). We will employ this procedure to demonstrate a simple and very useful relationship between the first passage time density and the Green's function for a broad class of additive skip-free processes. The processes are those in continuous time on the continuum $-\infty < x < \infty$, homogeneous in space and time, and having the skip-free property in the negative direction. Contained within this class are the homogeneous diffusion processes, the homogeneous Takács type processes with negative drift and positive increments (cf. Takács [15], Kendall [11]) and mixtures of the two types. The Green's function $G(x, t)$, a probability density in the generalized sense of L. Schwartz, defined by

$$G(x, t) = (d/dx)\Pr\{X(t) \leqq x \mid X(0) = 0\} \tag{1}$$

has the characteristic function

$$g(k, t) = \mathfrak{F}\{G(x, t)\} = \exp\{-Dk^2t - i\alpha kt - \nu t[1 - a^+(k)]\}. \tag{2}$$

In (2), D, α, and ν are non-negative constants, and

$$a^+(k) = \int_0^\infty e^{ikx}\, dF_A(x) \tag{3}$$

where $F_A(x)$ is the distribution governing the finite jumps. The characteristic function of (2) is seen to have infinitely divisible form (cf. Gnedenko and Kolmogorov, [5], Section 16).

For the process $X(t)$ commencing at $X(0) = x_0 > 0$, let τ be the first time at which $X(0) = 0$, and let the corresponding first passage time density (generalized) be denoted by $S(x_0, \tau)$. The basic result of interest is that this density takes the simple form

$$S(x_0, \tau) = (x_0/\tau)G(-x_0, \tau). \tag{4}$$

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