PROPERTIES OF GENERALIZED RAYLEIGH DISTRIBUTIONS

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1. Introduction. Some years ago we defined generalized Rayleigh processes [6], [7] and considered some of their many properties. Briefly, if x_i, $1 \leqq i \leqq n$, are Gaussian variates and r^2 is the sum of their squares, then we called r a generalized Rayleigh random variable. Besides the references in [6], [7] we have noted other investigations in this direction [2], [8]. In the present paper we wish to continue our study of Rayleigh distributions.

Our results exploit the methods employed in the theory of linear vector spaces and are of two types. The next three sections deal with explicit formulas; the last two sections with symbolic representations. In Section 2 we compute the joint p-dimensional Rayleigh distribution for a certain class of covariance matrices. The result is expressed in terms of modified Bessel functions of the first kind, [cf. (2.1)]. In Section 3 we compute the distribution of the inner product of two Gaussian vectors. The result is expressed in terms of a modified Bessel function of the second kind, [cf. (3.1)]. In Section 4 we compute the distribution of the difference of squares of norms of two Gaussian vectors. The result is expressed in terms of Whittaker functions, [cf. (4.1)]. Precise definitions and assumptions are made in the theorems leading to (2.1), (3.1) and (4.1).

The problem of computing the p-dimensional Rayleigh distribution in a useful form for arbitrary covariance matrices appears intractable. However we do obtain symbolic (operator) forms for the p-dimensional distribution for both the biased and unbiased cases, [cf. (5.1)]. These results seem to be of theoretical value in related investigations. In the final section, Section 6, we obtain a symbolic form for the density function of a Rayleigh variate when the variances of the Gaussian components are not necessarily equal, [cf. (6.1)].

We are indebted to the referee for pointing out certain additional references, as well as for making the observation that the theorems of Sections 2 and 3 could be approached by starting with the Wishart distribution.

2. The p-dimensional distribution. Let $Y_1, Y_2, \cdots, Y_n$ be p-dimensional column vectors which are independent and identically normally distributed with mean zero and covariance matrix M. Let X_k be an n-dimensional vector comprised of the kth components of the Y_j, $1 \leqq j \leqq n$. Then under certain restrictions on M we shall compute the joint p-dimensional distribution of the norms of the X_k, $1 \leqq k \leqq p$.

A word on notation: Primes will always denote transposes, that is, row vectors.

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