A COMBINATORIAL THEOREM FOR PARTIAL SUMS

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1. Introduction. Let $(x_1, \cdots, x_n)$ be a sequence of real numbers, $s_k = \sum_{j=1}^{k} x_j$ and $M_k = \max(s_1, \cdots, s_k)$. As usual let the superscript $^+$ mean maximize with zero. In a recent paper of M. Dwass [2], a theorem equivalent to the following is proved (generalizing a result of Kac [3] and Spitzer [4]):

Theorem. $\sum_\sigma (M_n^+ - M_{n-1}^+) = s_n^+$ *where* σ *ranges over all cyclic permutations of* $(x_1, \cdots, x_n)$.

In this note we give a generalization of this theorem. It will be seen that several recent results of L. Takács [5], [6] may be derived from this extension.

2. The basic theorems. We begin with a preliminary lemma. Let $X = (x_1, \cdots, x_n)$ be a sequence of real numbers. Let $m(X)$ denote the rth *largest* term of X (or zero if $r > n$) and let $m(X, y)$ abbreviate $m((x_1, \cdots, x_n, y))$.

Lemma. *If* $y \geqq 0$ *then*

$$m(X, y)^+ - m(X)^+ = m(X, y) - m(X, 0). \tag{1}$$

Proof. There are three cases:

(i) Suppose $m(X, y) \leqq 0$. Since $y \geqq 0$ then $m(X, 0) = m(X, y)$ and $m(X)^+ = m(X, y)^+ = 0$ and (1) follows.

(ii) Suppose $m(X, y) > 0$ and $m(X) > 0$. Then $m(X, y)^+ = m(X, y)$ and $m(X)^+ = m(X) = m(X, 0)$ and (1) follows.

(iii) Suppose $m(X, y) > 0$ and $m(X) \leqq 0$. Since $y \geqq 0$ then a moment's reflection shows that $m(X)^+ = 0 = m(X, 0)$. But we have $m(X, y)^+ = m(X, y)$ and so (1) follows.

This completes the proof.

Now denote the partial sum $\sum_{j=1}^{k} x_j$ by s_k. Suppose $1 \leqq r \leqq n$ and let $m_k = m((s_1, \cdots, s_k))$. Then we have

Theorem 1. $\sum_\sigma (m_n^+ - m_{n-1}^+) = s_n^+$ *where* σ *ranges over all cyclic permutations of* $(x_1, \cdots, x_n)$.

Proof. If $s_n < 0$ then the theorem is immediate since in this case $s_n^+ = 0 = m_n^+ - m_{n-1}^+$ for all permutations of the x_i. Assume $s_n \geqq 0$ and note that

$$\begin{aligned} m_n &= m((x_1, x_1 + x_2, \cdots, x_1 + \cdots + x_n)) \\ &= x_1 + m((0, x_2, x_2 + x_3, \cdots, x_2 + \cdots + x_n)). \end{aligned}$$

Therefore by the lemma

$$\begin{aligned} \sum_\sigma (m_n^+ - m_{n-1}^+) &= \sum_\sigma (m_n - m((x_1, x_1 + x_2, \cdots, x_1 + \cdots + x_{n-1}, 0))) \\ &= \sum_\sigma (x_1 + m((0, x_2, x_2 + x_3, \cdots, x_2 + \cdots + x_n)) \\ &\qquad - m((0, x_1, x_1 + x_2, \cdots, x_1 + \cdots + x_{n-1}))) \\ &= x_1 + \cdots + x_n = s_n \end{aligned}$$

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